Lecture 28

Antennas and Radiation and the Hertzian Dipole

In this lecture you will learn:

- **Generation of radiation by oscillating charges and currents**
- **Hertzian dipole antenna**

Maxwell's Equations and Radiation

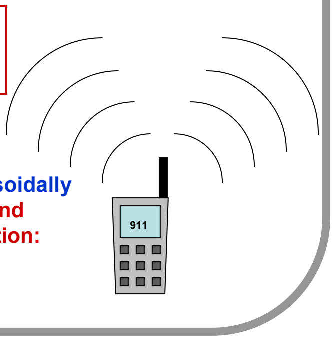
$$\nabla . \, \mu_o \, \vec{H}(\vec{r},t) = 0 \qquad \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \, \mu_o \vec{H}(\vec{r},t)}{\partial t}$$

$$\nabla . \, \varepsilon_o \, \vec{E}(\vec{r},t) = \rho(\vec{r},t) \qquad \nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \, \varepsilon_o \vec{E}(\vec{r},t)}{\partial t}$$

Time-varying currents as the "source" or the "driving term" for the wave equation:

$$\nabla \times \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \, \mu_o \nabla \times \vec{H}(\vec{r},t)}{\partial t} = -\frac{\partial \, \mu_o \vec{J}(\vec{r},t)}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla \times \nabla \times \vec{E}(\vec{r},t) + \frac{1}{c^2} \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} = -\frac{\partial \, \mu_o \vec{J}(\vec{r},t)}{\partial t}}$$



Maxwell's equation predict outgoing radiation from sinusoidally time-varying currents (and charges - recall that current and charge densities are related through the continuity equation:

$$\nabla . \, \vec{J}(\vec{r},t) + \frac{\partial \rho(\vec{r},t)}{\partial t} = 0$$

Electro- and Magneto-quasistatics and Potentials

Electroquasistatics

$$\nabla . \, \varepsilon_o \, \vec{E}(\vec{r},t) = \rho(\vec{r},t)$$

$$\nabla \times \vec{E}(\vec{r},t) = 0$$

Since:

$$\nabla \times \vec{E}(\vec{r},t) = 0$$

One could write:

$$\vec{E}(\vec{r},t) = -\nabla \phi(\vec{r},t)$$

Scalar potential

Magnetoquasistatics

$$\nabla . \, \mu_o \, \vec{H}(\vec{r},t) = 0$$

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t)$$

Since:

$$\nabla . \, \mu_o \, \vec{H}(\vec{r},t) = 0$$

One could write:

$$\mu_o \, \vec{H}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t)$$

Vector potential

Electrodynamics and Potentials - I

$$\nabla . \, \mu_o \, \vec{H}(\vec{r},t) = 0 \qquad \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \, \mu_o \vec{H}(\vec{r},t)}{\partial t}$$

$$\nabla . \, \varepsilon_o \, \vec{E}(\vec{r},t) = \rho(\vec{r},t) \qquad \nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \, \varepsilon_o \vec{E}(\vec{r},t)}{\partial t}$$

Vector Potential

One still has:

$$\nabla . \, \mu_o \, \vec{H}(\vec{r},t) = 0$$

Therefore, one can still introduce a vector potential:

$$\mu_o \, \vec{H}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t)$$

Faraday's law then becomes:

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \, \mu_o \vec{H}(\vec{r},t)}{\partial t}$$

$$\Rightarrow \quad \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \, \nabla \times \vec{A}(\vec{r},t)}{\partial t}$$

$$\Rightarrow \quad \nabla \times \left[\vec{E}(\vec{r},t) + \frac{\partial \, \vec{A}(\vec{r},t)}{\partial t} \right] = 0$$

Electrodynamics and Potentials - II

Scalar Potential

Since: $$\nabla \times \left[\vec{E}(\vec{r},t) + \frac{\partial\, \vec{A}(\vec{r},t)}{\partial t} \right] = 0$$

One may introduce a scalar potential as follows:

$$\vec{E}(\vec{r},t) + \frac{\partial\, \vec{A}(\vec{r},t)}{\partial t} = -\nabla \phi(\vec{r},t)$$

$$\Rightarrow \quad \vec{E}(\vec{r},t) = -\frac{\partial\, \vec{A}(\vec{r},t)}{\partial t} - \nabla \phi(\vec{r},t)$$

Using the vector and scalar potentials, the expressions for E-field and H-field become:

$$\boxed{\begin{aligned} &\mu_o \vec{H}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t) \\ &\vec{E}(\vec{r},t) = -\frac{\partial\, \vec{A}(\vec{r},t)}{\partial t} - \nabla \phi(\vec{r},t) \end{aligned}}$$

Choosing a "Gauge" in Electromagnetism

Non-uniqueness of the vector potential

The vector potential *A* is not unique – only the curl of the vector potential is a well defined quantity (i.e. the B-field):

$$\mu_o \vec{H}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t)$$

Demonstration: **suppose we change the vector potential - such that the new vector potential is the old vector potential plus the gradient of some arbitrary function**

$$\vec{A}_{new}(\vec{r},t) = \vec{A}(\vec{r},t) + \nabla \psi(\vec{r},t)$$

Then:

$$\left. \begin{aligned} &\nabla \times \vec{A}_{new}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t) + \cancelto{0}{\nabla \times \nabla \psi(\vec{r},t)} \\ \Rightarrow \quad &\nabla \times \vec{A}_{new}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t) \end{aligned} \right\}$$

The new vector potential is just as good as it will give the same B-field

Making the vector potential unique

A vector field can be uniquely specified (up to a constant) by specifying the value of its curl and its divergence

To make the vector potential *A* unique, one needs to fix the value of its divergence – a process that usually goes by the names "gauge fixing" or "fixing the gauge" or "choosing a gauge"

$$\nabla . \vec{A}(\vec{r},t) = ??$$

Gauges in Electromagnetism

Coulomb Gauge:

$$\boxed{\nabla . \vec{A}(\vec{r},t) = 0}$$

- This gauge is commonly used in electro- and magnetoquasistatics
- This gauge is not commonly used in electrodynamics

Lorentz Gauge:

$$\boxed{\nabla . \vec{A}(\vec{r},t) = -\frac{1}{c^2}\frac{\partial \phi(\vec{r},t)}{\partial t}} \longrightarrow \left\{ \right.$$

Relates divergence of the vector potential to the time derivative of the scalar potential

- This gauge is commonly used in electrodynamics

Vector Potential Wave Equation

Using the vector and scalar potentials, the expressions for E-field and H-field were:

$$\mu_o \vec{H}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t) \qquad \vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} - \nabla \phi(\vec{r},t)$$

Ampere's Law becomes:

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \varepsilon_o \vec{E}(\vec{r},t)}{\partial t}$$

$$\Rightarrow \quad \nabla \times \nabla \times \vec{A}(\vec{r},t) = \mu_o \vec{J}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} - \frac{1}{c^2}\nabla\left[\frac{\partial \phi(\vec{r},t)}{\partial t}\right]$$

Remembering that:

$$\nabla \times \nabla \times \vec{A}(\vec{r},t) = \nabla\left[\nabla . \vec{A}(\vec{r},t)\right] - \nabla^2 \vec{A}(\vec{r},t) \qquad \text{and} \qquad \nabla . \vec{A}(\vec{r},t) = -\frac{1}{c^2}\frac{\partial \phi(\vec{r},t)}{\partial t}$$

We finally get:

$$\boxed{\nabla^2 \vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_o \vec{J}(\vec{r},t)}$$

This is the vector potential wave equation with current as the driving term

Scalar Potential Wave Equation

Using the vector and scalar potentials, the expressions for E-field and H-field were:

$$\mu_o \vec{H}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t) \qquad \vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} - \nabla \phi(\vec{r},t)$$

Gauss' Law becomes:

$$\nabla . \varepsilon_o \vec{E}(\vec{r},t) = \rho(\vec{r},t)$$

$$\Rightarrow \quad -\nabla^2 \phi(\vec{r},t) - \frac{\partial \nabla . A(\vec{r},t)}{\partial t} = \frac{\rho(\vec{r},t)}{\varepsilon_o}$$

Remembering that:

$$\nabla . \vec{A}(\vec{r},t) = -\frac{1}{c^2}\frac{\partial \phi(\vec{r},t)}{\partial t}$$

We finally get:

$$\boxed{\nabla^2 \phi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \phi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_o}}$$

This is the scalar potential wave equation with charge as the driving term

Scalar and Vector Potential Wave Equations

Given any arbitrary time-dependent charge and current density distributions one can solve these two wave equations to get the potentials:

$$\boxed{\nabla^2 \phi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \phi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_o}}$$

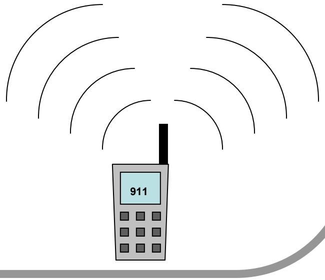
Scalar potential wave equation

$$\boxed{\nabla^2 \vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_o \vec{J}(\vec{r},t)}$$

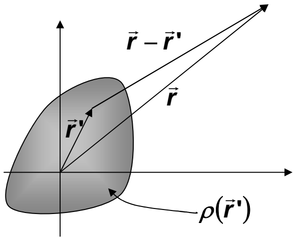
Vector potential wave equation

And then find the E- and H-fields using:

$$\boxed{\begin{aligned}&\mu_o \vec{H}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t)\\ &\vec{E}(\vec{r},t) = -\frac{\partial \vec{A}(\vec{r},t)}{\partial t} - \nabla \phi(\vec{r},t)\end{aligned}}$$



Superposition Integral Solution of the Scalar Wave Equation



We know that Poisson equation in electrostatics:

$$\nabla^2\phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o}$$

Has the solution:

$$\phi(\vec{r}) = \iiint \frac{\rho(\vec{r}\,')}{4\pi\,\varepsilon_o\,|\vec{r}-\vec{r}\,'|}\,dv'$$

The wave equation (which looks somewhat similar to the Poisson equation):

$$\nabla^2\phi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\phi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_o}$$

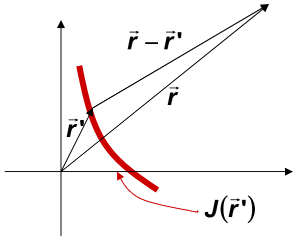
Has the solution:

$$\phi(\vec{r},t) = \iiint \frac{\rho\left(\vec{r}\,',t-|\vec{r}-\vec{r}\,'|/c\right)}{4\pi\,\varepsilon_o\,|\vec{r}-\vec{r}\,'|}\,dv' \longrightarrow$$

Retarded Potential:

- The potential at the observation point at any time corresponds to the charge at the source point at an earlier time
- Electromagnetic *disturbances* travel at the speed *c*

Superposition Integral Solution of the Vector Wave Equation



We know that the vector Poisson equation in magnetostatics:

$$\nabla^2\vec{A}(\vec{r}) = -\mu_o\vec{J}(\vec{r})$$

Has the solution:

$$\vec{A}(\vec{r}) = \iiint \frac{\mu_o\,\vec{J}(\vec{r}\,')}{4\pi\,|\vec{r}-\vec{r}\,'|}\,dv'$$

The wave equation (which looks somewhat similar to the vector Poisson equation):

$$\nabla^2\vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2\vec{A}(\vec{r},t)}{\partial t^2} = -\mu_o\vec{J}(\vec{r},t)$$

Has the solution:

$$\vec{A}(\vec{r},t) = \iiint \frac{\mu_o\,\vec{J}\left(\vec{r}\,',t-|\vec{r}-\vec{r}\,'|/c\right)}{4\pi\,|\vec{r}-\vec{r}\,'|}\,dv' \longrightarrow$$

Retarded Potential:

- The potential at the observation point at any time corresponds to the current at the source point at an earlier time
- Electromagnetic *disturbances* travel at the speed *c*

Time-Harmonic Fields and Complex Wave Equations

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}(\vec{r})\, e^{j\omega t}\right] \qquad \vec{H}(\vec{r},t) = \mathrm{Re}\left[\vec{H}(\vec{r})\, e^{j\omega t}\right]$$

$$\vec{A}(\vec{r},t) = \mathrm{Re}\left[\vec{A}(\vec{r})\, e^{j\omega t}\right] \qquad \phi(\vec{r},t) = \mathrm{Re}\left[\phi(\vec{r})\, e^{j\omega t}\right]$$

$$\rho(\vec{r},t) = \mathrm{Re}\left[\rho(\vec{r})\, e^{j\omega t}\right] \qquad \vec{J}(\vec{r},t) = \mathrm{Re}\left[\vec{J}(\vec{r})\, e^{j\omega t}\right]$$

Assuming time harmonic currents, charges, and fields, the wave equations:

$$\nabla^2 \vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_o \vec{J}(\vec{r},t)$$

$$\nabla^2 \phi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \phi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_o}$$

become:

$$\left.\begin{aligned} &\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}) = -\mu_o \vec{J}(\vec{r}) \\ &\nabla^2 \phi(\vec{r}) + k^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o} \end{aligned}\right\} \longrightarrow k = \frac{\omega}{c}$$

Superposition Integral Solutions of the Complex Wave Equations

For time harmonic signals:

$$\vec{A}(\vec{r},t) = \mathrm{Re}\left[\vec{A}(\vec{r})\, e^{j\omega t}\right] \qquad \phi(\vec{r},t) = \mathrm{Re}\left[\phi(\vec{r})\, e^{j\omega t}\right]$$

$$\rho(\vec{r},t) = \mathrm{Re}\left[\rho(\vec{r})\, e^{j\omega t}\right] \qquad \vec{J}(\vec{r},t) = \mathrm{Re}\left[\vec{J}(\vec{r})\, e^{j\omega t}\right]$$

The solutions to the complex wave equations are found as follows:

$$\phi(\vec{r},t) = \iiint \frac{\rho\left(\vec{r}\,',t-\left|\vec{r}-\vec{r}\,'\right|/c\right)}{4\pi\,\varepsilon_o \left|\vec{r}-\vec{r}\,'\right|}\, dv' \Longrightarrow \boxed{\phi(\vec{r}) = \iiint \frac{\rho(\vec{r}\,')}{4\pi\,\varepsilon_o \left|\vec{r}-\vec{r}\,'\right|}\, e^{-jk\left|\vec{r}-\vec{r}\,'\right|}\, dv'}$$

Where we have used:

$$\rho\left(\vec{r}\,',t-\left|\vec{r}-\vec{r}\,'\right|/c\right) = \mathrm{Re}\left[\rho(\vec{r}\,')e^{j\omega\left(t-\left|\vec{r}-\vec{r}\,'\right|/c\right)}\right] = \mathrm{Re}\left[\rho(\vec{r}\,')e^{-j\omega\left|\vec{r}-\vec{r}\,'\right|/c}\;\; e^{j\omega t}\right]$$

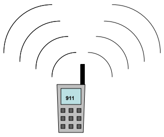
And:

$$\vec{A}(\vec{r},t) = \iiint \frac{\mu_o\, \vec{J}\left(\vec{r}\,',t-\left|\vec{r}-\vec{r}\,'\right|/c\right)}{4\pi \left|\vec{r}-\vec{r}\,'\right|}\, dv' \Longrightarrow \boxed{\vec{A}(\vec{r}) = \iiint \frac{\mu_o\, \vec{J}(\vec{r}\,')}{4\pi \left|\vec{r}-\vec{r}\,'\right|}\, e^{-jk\left|\vec{r}-\vec{r}\,'\right|}\, dv'}$$

Where we have used:

$$\vec{J}\left(\vec{r}\,',t-\left|\vec{r}-\vec{r}\,'\right|/c\right) = \mathrm{Re}\left[\vec{J}(\vec{r}\,')e^{j\omega\left(t-\left|\vec{r}-\vec{r}\,'\right|/c\right)}\right] = \mathrm{Re}\left[\vec{J}(\vec{r}\,')e^{-j\omega\left|\vec{r}-\vec{r}\,'\right|/c}\;\; e^{j\omega t}\right]$$

Wave Equations and Methods of Solution



Suppose we need to find the radiation emitted by some collection of sinusoidally time varying charges and currents

Method 1

(1) We can solve these two equations:

$$\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}) = -\mu_o \vec{J}(\vec{r}) \qquad \nabla^2 \phi(\vec{r}) + k^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_o}$$

(2) And then find the E-field and the H-field through:

$$\mu_o \vec{H}(\vec{r}) = \nabla \times \vec{A}(\vec{r}) \qquad \vec{E}(\vec{r}) = -j\omega \vec{A}(\vec{r}) - \nabla \phi(\vec{r}, t)$$

Method 2

(1) We can solve just one equation:

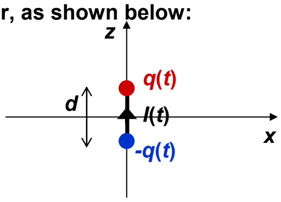
$$\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}) = -\mu_o \vec{J}(\vec{r})$$

(2) And then find the H-field and the E-field through:

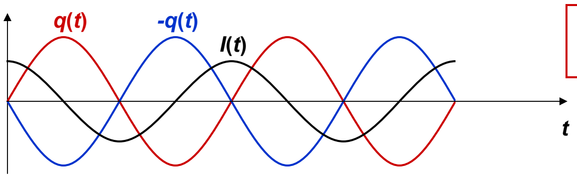
$$\mu_o \vec{H}(\vec{r}) = \nabla \times \vec{A}(\vec{r}) \qquad \nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\omega \varepsilon_o \vec{E}(\vec{r})$$

Hertzian Dipole Antenna - I

- A Hertzian dipole is one of the simplest radiating elements for which analytical solutions for the fields can be obtained
- A Hertzian dipole consists of two equal and opposite ± charge reservoirs located at a distance *d* from each other, as shown below:

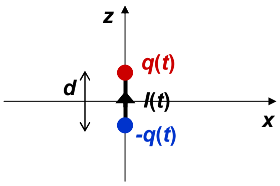


- The two charge reservoirs are electrically connected and a sinusoidal current *I*(*t*) flows between them
- Consequently, the charge in the reservoirs also changes sinusoidally:



$$\boxed{I(t) = \frac{dq(t)}{dt}}$$

Hertzian Dipole Antenna - II



Suppose we could write a current density $\vec{J}(\vec{r})$ for the Hertzian dipole

Then:

$$\iiint \vec{J}(\vec{r})\, dv = \iiint \vec{J}(\vec{r})\, dx\, dy\, dz = \hat{z}\, I\, d$$

- **The integral represents the total "weight" or "strength" of the dipole**
- **If the size of the dipole is much smaller than the wavelength then one may write:**

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\; \delta(x)\delta(y)\,\delta(z)$$

$$= \hat{z}\, I\, d\; \delta^3(\vec{r})$$

The above expression will give the same strength for the dipole, i.e.

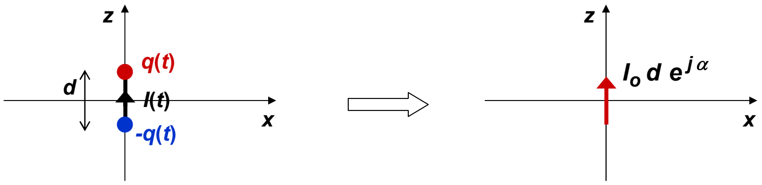
$$\iiint \vec{J}(\vec{r})\, dv = \hat{z}\, I\, d$$

Check units:

$$\left.\begin{array}{l} \delta^3(\vec{r}) \rightarrow 1/\mathrm{m}^3 \\ I \rightarrow \mathrm{Amp} \\ d \rightarrow \mathrm{m} \end{array}\right\} \Rightarrow \quad \vec{J}(\vec{r}) \rightarrow \mathrm{Amp}/\mathrm{m}^2$$

Hertzian Dipole Antenna - III

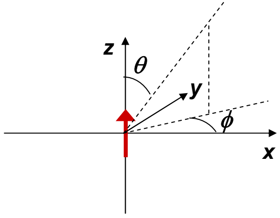
- **A Hertzian dipole is represented by an arrow whose direction indicates the positive direction of the current and also the orientation of the dipole in space:**



Example: If $I(t) = I_o \cos(\omega t + \alpha)$ then: $\vec{J}(\vec{r}) = \hat{z}\, I_o\, d\, e^{j\alpha}\, \delta^3(\vec{r})$

Radiation Emitted by a Hertzian Dipole - I

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r})$$



Working in spherical coordinates

Need to solve:

$$\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}) = -\mu_o \vec{J}(\vec{r})$$

Use the superposition integral:

$$\vec{A}(\vec{r}) = \iiint \frac{\mu_o}{4\pi} \frac{\vec{J}(\vec{r}\,')}{\left|\vec{r}-\vec{r}\,'\right|} e^{-j\,k\left|\vec{r}-\vec{r}\,'\right|}\, dv'$$

$$\Rightarrow \quad \vec{A}(\vec{r}) = \hat{z}\, \frac{\mu_o\, Id}{4\pi\, \left|\vec{r}\right|} e^{-j\,k\left|\vec{r}\right|} = \hat{z}\, \frac{\mu_o\, Id}{4\pi\, r} e^{-j\,k\,r}$$

$$\Rightarrow \quad \vec{A}(\vec{r}) = \left[\hat{r}\cos(\theta) - \hat{\theta}\sin(\theta)\right] \frac{\mu_o\, Id}{4\pi\, r} e^{-j\,k\,r}$$

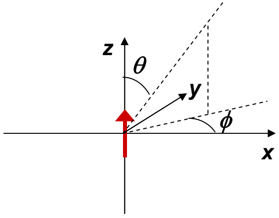
Finding the H-field:

$$\mu_o \vec{H}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$$

$$\Rightarrow \quad \vec{H}(\vec{r}) = \hat{\phi}\, \frac{j\,k\,Id}{4\pi\, r} e^{-j\,k\,r} \left[1 + \frac{1}{j\,k\,r}\right] \sin(\theta)$$

Radiation Emitted by a Hertzian Dipole - II

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r})$$



Working in spherical coordinates

H-field was:

$$\vec{H}(\vec{r}) = \hat{\phi}\, \frac{j\,k\,Id}{4\pi\, r} e^{-j\,k\,r} \left[1 + \frac{1}{j\,k\,r}\right] \sin(\theta)$$

Finding the E-field:

Use Ampere's Law: $\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\omega\,\varepsilon_o \vec{E}(\vec{r})$

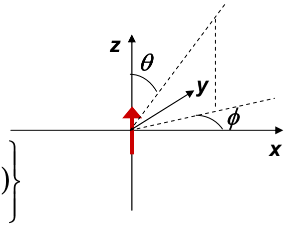
Away from the dipole the current density is zero, therefore:

$$\vec{E}(\vec{r}) = \frac{1}{j\omega\,\varepsilon_o} \nabla \times \vec{H}(\vec{r})$$

$$\vec{E}(\vec{r}) = \eta_o \frac{j\,k\,Id}{4\pi\, r} e^{-j\,k\,r} \left\{ \hat{r} \left[\frac{1}{j\,k\,r} + \left(\frac{1}{j\,k\,r}\right)^2\right] 2\cos(\theta) + \hat{\theta}\left[1 + \frac{1}{j\,k\,r} + \left(\frac{1}{j\,k\,r}\right)^2\right] \sin(\theta) \right\}$$

Near-Fields of a Hertzian Dipole - I

$$\vec{J}(\vec{r}) = \hat{z}\, I\, d\, \delta^3(\vec{r})$$



$$\vec{H}(\vec{r}) = \hat{\phi}\,\frac{j\,k\,Id}{4\pi\,r}\,e^{-j\,k\,r}\left[1+\frac{1}{j\,k\,r}\right]\sin(\theta)$$

$$\vec{E}(\vec{r}) = \eta_o\,\frac{j\,k\,Id}{4\pi\,r}\,e^{-j\,k\,r}\left\{\hat{r}\left[\frac{1}{j\,k\,r}+\left(\frac{1}{j\,k\,r}\right)^2\right]2\cos(\theta)+\hat{\theta}\left[1+\frac{1}{j\,k\,r}+\left(\frac{1}{j\,k\,r}\right)^2\right]\sin(\theta)\right\}$$

Near-field is the field close to the dipole where: $kr \ll 1$
(or more accurately where: $d \ll r \ll \lambda/2\pi$)

$$\vec{E}_{nf}(\vec{r}) = \eta_o\,\frac{Id}{4\pi\,jk\,r^3}\left[\,\hat{r}\;\;2\cos(\theta)+\hat{\theta}\,\sin(\theta)\,\right] = \frac{qd}{4\pi\,\varepsilon_o\,r^3}\left[\,\hat{r}\;\;2\cos(\theta)+\hat{\theta}\,\sin(\theta)\,\right]$$

$$\vec{H}_{nf}(\vec{r}) = \hat{\phi}\,\frac{Id}{4\pi\,r^2}\sin(\theta) = \hat{\phi}\;j\omega\,\frac{qd}{4\pi\,r^2}\sin(\theta)$$

E-field and H-field are 90-degrees out of phase in the near-field

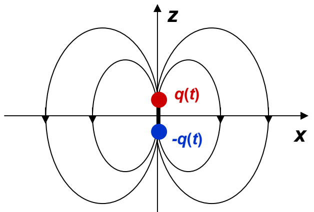
$$\left\{\; I(t) = \frac{dq(t)}{dt} \quad \Rightarrow \quad I = j\omega\,q \right.$$

Near-Fields of a Hertzian Dipole - II

Near fields of a Hertzian-dipole are quasistatic in nature

1) E-field corresponds to the instantaneous value of the charge dipole

$$\vec{E}_{nf}(\vec{r},t) = \frac{q(t)\,d}{4\pi\,\varepsilon_o\,r^3}\left[\,\hat{r}\;\;2\cos(\theta)+\hat{\theta}\,\sin(\theta)\,\right]$$



$$I(t) = \frac{dq(t)}{dt}$$

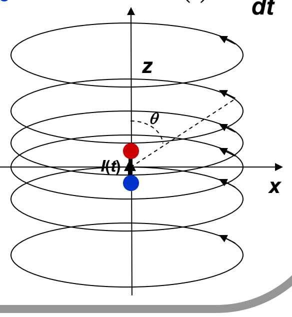
2) H-field corresponds to the instantaneous value of the current and can be obtained from the Biot-Savart law

$$\vec{H}_{nf}(\vec{r},t) = \hat{\phi}\,\frac{I(t)\,d}{4\pi\,r^2}\sin(\theta)$$

Proof:

$$\vec{H}_{nf}(\vec{r},t) = \frac{I(t)}{4\pi}\int\frac{d\vec{s}^{\,'}\times\hat{n}_{\vec{s}'\to\vec{r}}}{\left|\vec{r}-\vec{s}'\right|^2}$$

$$= \frac{I(t)}{4\pi}\int\frac{\hat{z}\,dz'\times\hat{r}}{r^2} = \hat{\phi}\,\frac{I(t)\,d}{4\pi\,r^2}\sin(\theta)$$

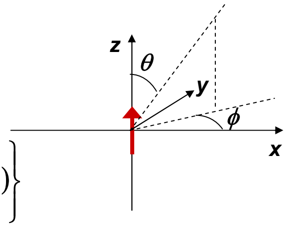


Far-Fields of a Hertzian Dipole

$$\vec{H}(\vec{r}) = \hat{\phi}\,\frac{j\,k\,Id}{4\pi\,r}\,e^{-j\,k\,r}\left[1+\frac{1}{j\,k\,r}\right]\sin(\theta)$$

$$\vec{E}(\vec{r}) = \eta_o\,\frac{j\,k\,Id}{4\pi\,r}\,e^{-j\,k\,r}\left\{\hat{r}\left[\frac{1}{j\,k\,r}+\left(\frac{1}{j\,k\,r}\right)^2\right]2\cos(\theta)+\right.$$
$$\left.\hat{\theta}\left[1+\frac{1}{j\,k\,r}+\left(\frac{1}{j\,k\,r}\right)^2\right]\sin(\theta)\right\}$$

$$\vec{J}(\vec{r}) = \hat{z}\,I\,d\,\delta^3(\vec{r})$$



Far-field is the field far away from the dipole where: $kr >> 1$

(or more accurately where: $d << \lambda/2\pi << r$)

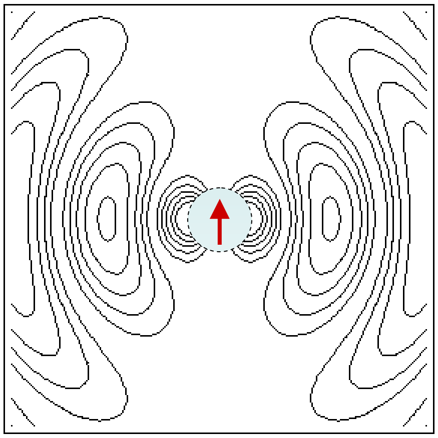
$$\vec{H}_{ff}(\vec{r}) = \hat{\phi}\,\frac{j\,k\,Id}{4\pi\,r}\,e^{-j\,k\,r}\sin(\theta)$$

$$\vec{E}_{ff}(\vec{r}) = \hat{\theta}\,\frac{j\,\eta_o\,k\,Id}{4\pi\,r}\,e^{-j\,k\,r}\sin(\theta)$$

E-field and H-field are in phase in the far-field

Radiation Emitted by a Hertzian Dipole

$\vec{E}(\vec{r},t)$



Near field region
($r << \lambda/2\pi$)

$$\vec{E}(\vec{r}) = \eta_o\,\frac{j\,k\,Id}{4\pi\,r}\,e^{-j\,k\,r}\left\{\hat{r}\left[\frac{1}{j\,k\,r}+\left(\frac{1}{j\,k\,r}\right)^2\right]2\cos(\theta)+\hat{\theta}\left[1+\frac{1}{j\,k\,r}+\left(\frac{1}{j\,k\,r}\right)^2\right]\sin(\theta)\right\}$$