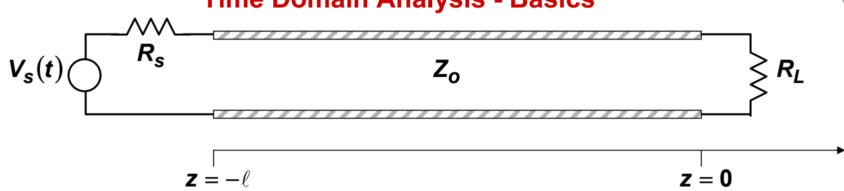
Lecture 24

Time Domain Analysis of Transmission Lines

In this lecture you will learn:

- **Time domain analysis of transmission lines**
- **Transients in transmission lines**

Time Domain Analysis - Basics



Question: How does one handle transmission lines for signals that are NOT time harmonic and when one is NOT dealing with the sinusoidal steady state?

a) **First thing to realize is that the notion of complex impedance has meaning only for the sinusoidal steady state**

b) **For an arbitrary source voltage $V_s(t)$, one needs to work in the time domain and start from the basic time-domain equations:**

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t}$$

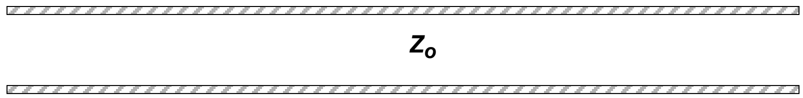
$$\longrightarrow$$

$$\frac{\partial^2 V(z,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 V(z,t)}{\partial t^2}$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 I(z,t)}{\partial t^2}$$

$$v = \frac{1}{\sqrt{LC}}$$

Time Domain Analysis - Basics

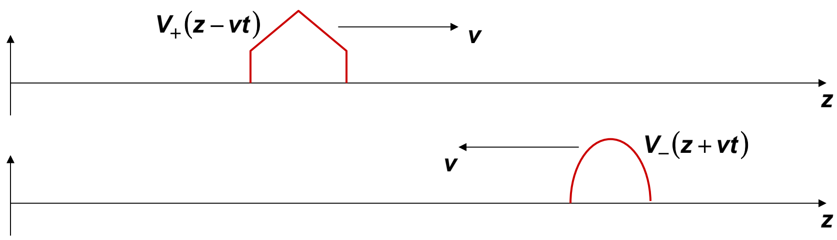


The equation: $\dfrac{\partial^2 V(z,t)}{\partial z^2} = \dfrac{1}{v^2}\dfrac{\partial^2 V(z,t)}{\partial t^2}$ $v = \dfrac{1}{\sqrt{LC}}$

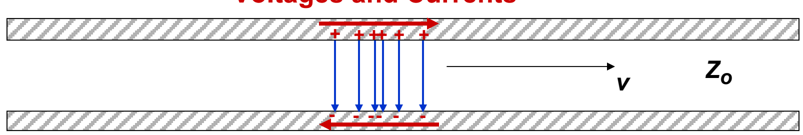
Has forward moving solutions of the form: $V(z,t) = V_+(z - vt)$

And backward moving solutions of the form: $V(z,t) = V_-(z + vt)$

Examples:



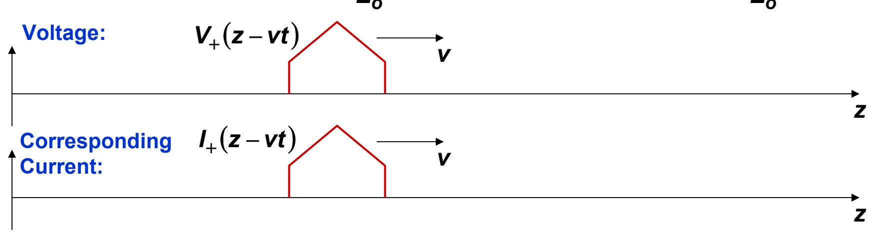
Voltages and Currents



The current is related to the voltage and satisfies: $\dfrac{\partial V(z,t)}{\partial z} = -L\dfrac{\partial I(z,t)}{\partial t}$

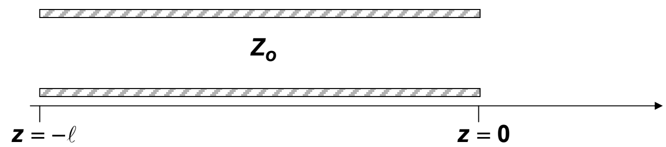
And this: $\dfrac{\partial I(z,t)}{\partial z} = -C\dfrac{\partial V(z,t)}{\partial t}$

The solution is: $I_+(z - vt) = \dfrac{V_+(z - vt)}{Z_o}$ and $I_-(z + vt) = -\dfrac{V_-(z + vt)}{Z_o}$



Current is proportional to voltage since higher voltage means more surface charges and more surface charges mean more current flow

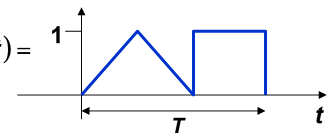
Visualizing Propagation



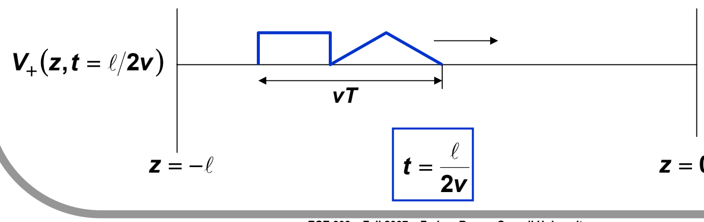
Forward moving solutions are of the form: $V_+(z - vt)$

And backward moving solutions are of the form: $V_-(z + vt)$

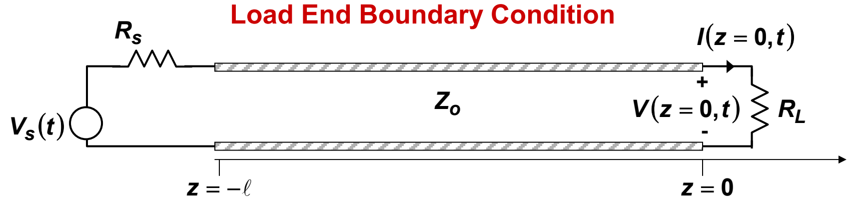
So suppose somebody tells you that at $z = -\ell$: $V_+(-\ell, t) =$



Then what is the forward moving voltage on the line at $t = \ell / 2v$?



Load End Boundary Condition



Load end boundary condition:

For all time t we must have:

$V(z=0,t) = I(z=0,t)R_L$

$$V(z=0,t) = V_+(z=0,t) + V_-(z=0,t)$$

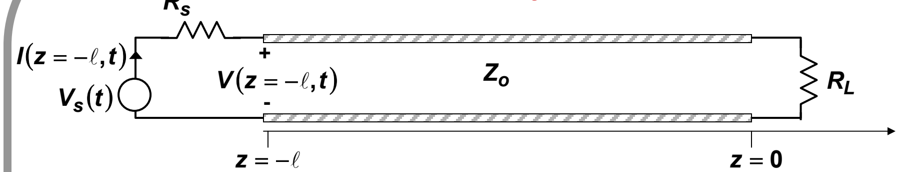
$$I(z=0,t) = I_+(z=0,t) + I_-(z=0,t) = \frac{V_+(z=0,t)}{Z_o} - \frac{V_-(z=0,t)}{Z_o}$$

Substitute these in this to get:

$$\Rightarrow \quad V_-(z=0,t) = V_+(z=0,t)\Gamma_L$$

$$\Gamma_L = \frac{R_L/Z_o - 1}{R_L/Z_o + 1}$$

Source End Boundary Condition



Source end boundary condition:

For all time t we must have:

$$V_s(t) = I(z=-\ell,t)R_s + V(z=-\ell,t)$$

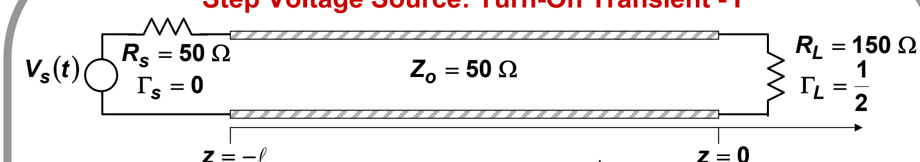
$$\begin{cases} V(z=-\ell,t) = V_+(z=-\ell,t) + V_-(z=-\ell,t) \\ I(z=-\ell,t) = I_+(z=-\ell,t) + I_-(z=-\ell,t) \\ \quad = \dfrac{V_+(z=-\ell,t)}{Z_o} - \dfrac{V_-(z=-\ell,t)}{Z_o} \end{cases}$$

Substitute these in this to get:

$$\Rightarrow \quad V_+(z=-\ell,t) = V_-(z=-\ell,t)\Gamma_S + V_s(t)\frac{Z_o}{R_s+Z_o}$$

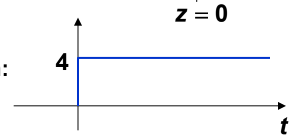
$$\left\{ \Gamma_S = \frac{R_S/Z_o - 1}{R_S/Z_o + 1} \right.$$

Step Voltage Source: Turn-On Transient - I



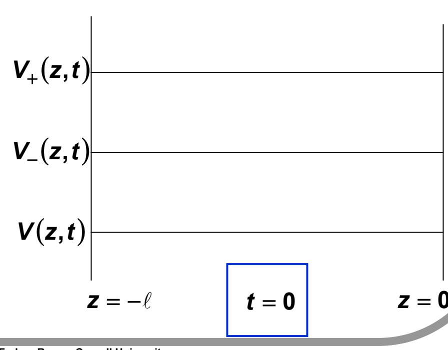
Suppose the source voltage is a step function:

$$V_s(t) = 4\,u(t)$$

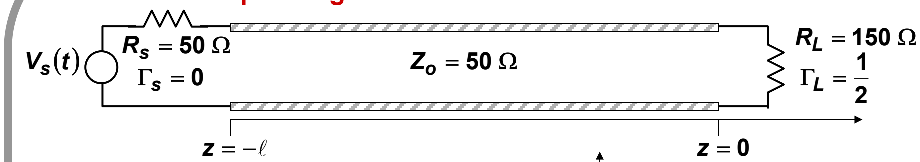


$$V_+(z=-\ell,t) = V_-(z=-\ell,t)\Gamma_S + V_s(t)\frac{Z_o}{R_s+Z_o}$$

$$\Rightarrow \quad V_+(z=-\ell,t>0) = 2$$

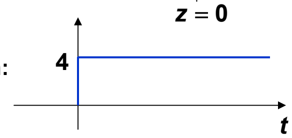


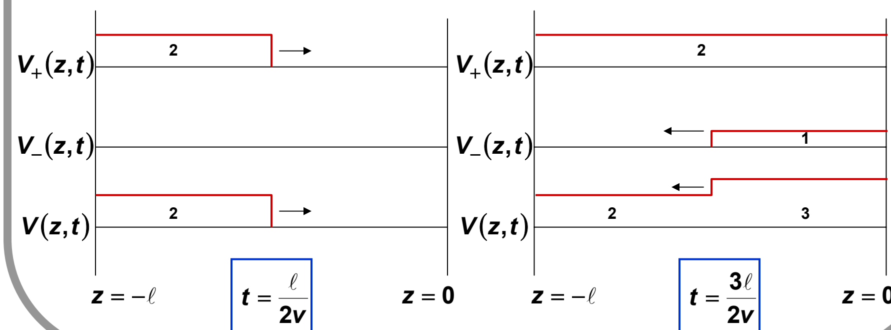
Step Voltage Source: Turn-On Transient - II



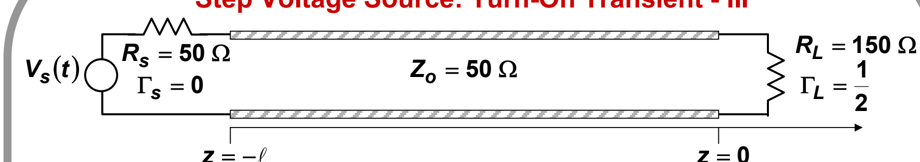
Suppose the source voltage is a step function:

$$V_s(t) = 4\,u(t)$$



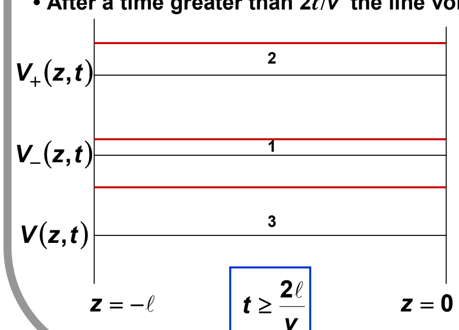


Step Voltage Source: Turn-On Transient - III



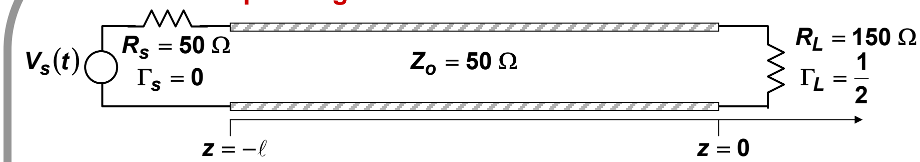
- The wave reflected back from the load end does not suffer a reflection at the source end since the source impedance is matched to the line impedance

- After a time greater than $2\ell/v$ the line voltage is at constant 3 Volts



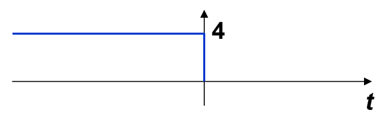
$$V_+(z=-\ell,t) = V_-(z=-\ell,t)\Gamma_s + V_s(t)\frac{Z_o}{R_s + Z_o}$$

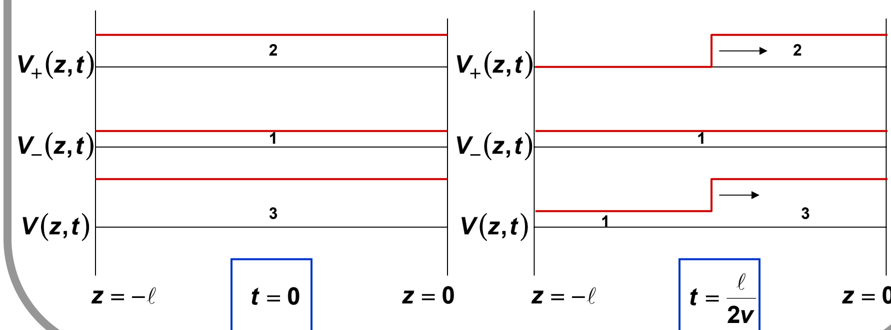
Step Voltage Source: Turn-Off Transient - I



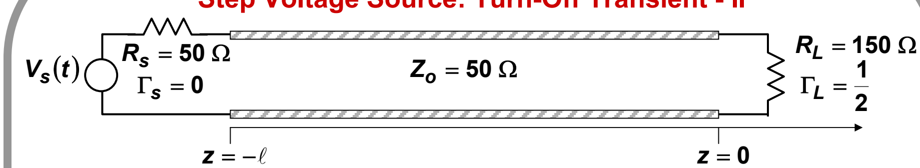
Suppose the source voltage has been at 4 Volts for a long long time – but is shut off at time $t = 0$

$$V_s(t) = 4[1 - u(t)]$$

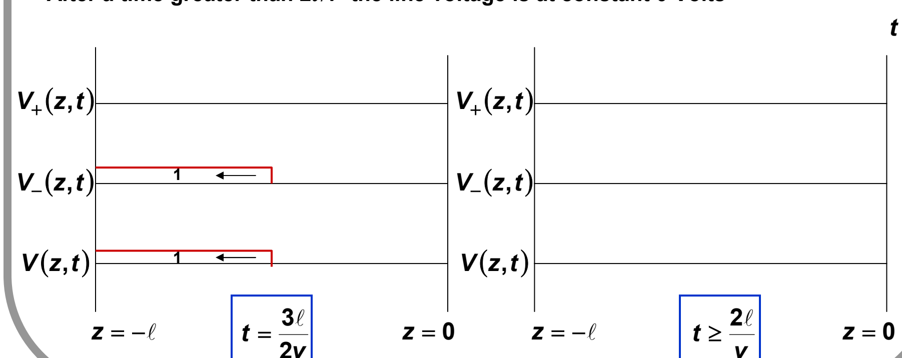




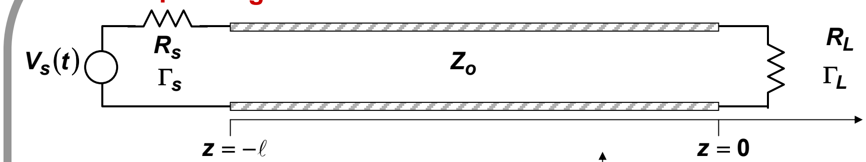
Step Voltage Source: Turn-Off Transient - II



- After a time greater than $2\ell/v$ the line voltage is at constant 0 Volts

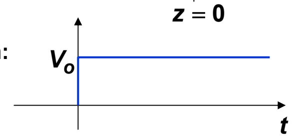


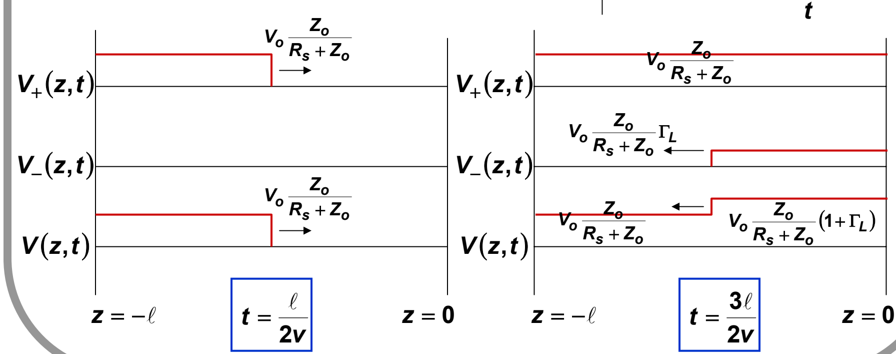
Step Voltage Source: Turn-On Transient – General Case



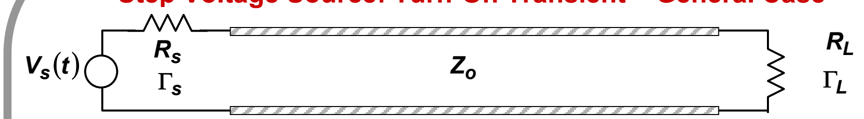
Suppose the source voltage is a step function:

$$V_s(t)=V_o\,u(t)$$





Step Voltage Source: Turn-On Transient – General Case



- The waves will keep bouncing forever
- But the net voltage on the line will slowly converge to the value one would expect in DC operation

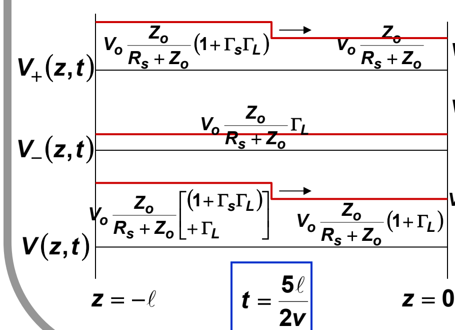
$$V_+(z,\infty)=V_o\frac{Z_o}{R_s+Z_o}\left[1+\Gamma_s\Gamma_L+(\Gamma_s\Gamma_L)^2+(\Gamma_s\Gamma_L)^3+........\right]$$

$$V_+(z,\infty)=V_o\frac{Z_o}{R_s+Z_o}\left[\frac{1}{1-\Gamma_s\Gamma_L}\right]$$

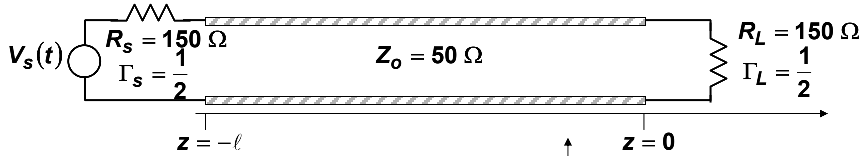
$$V_-(z,\infty)=V_o\frac{Z_o}{R_s+Z_o}\Gamma_L\left[1+\Gamma_s\Gamma_L+(\Gamma_s\Gamma_L)^2+(\Gamma_s\Gamma_L)^3+........\right]$$

$$V_-(z,\infty)=V_o\frac{Z_o}{R_s+Z_o}\left[\frac{\Gamma_L}{1-\Gamma_s\Gamma_L}\right]$$

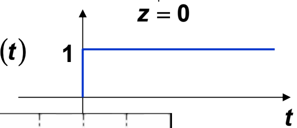
$$V(z,\infty)=V_+(z,\infty)+V_-(z,\infty)$$
$$=V_o\frac{Z_o}{R_s+Z_o}\frac{1+\Gamma_L}{1-\Gamma_s\Gamma_L}$$
$$=V_o\frac{R_L}{R_s+R_L}$$

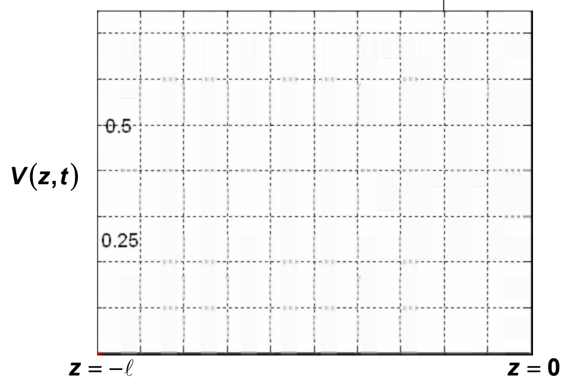


Step Voltage Source: Turn-On Transient

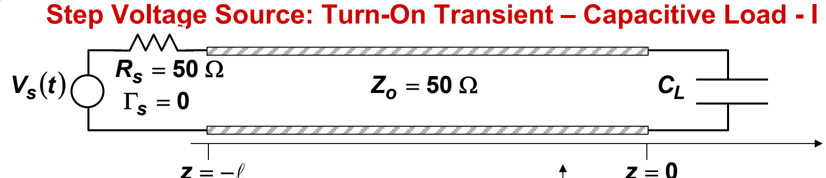


Source voltage is a step function: $V_s(t) = 1u(t)$

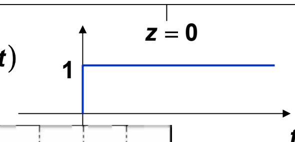


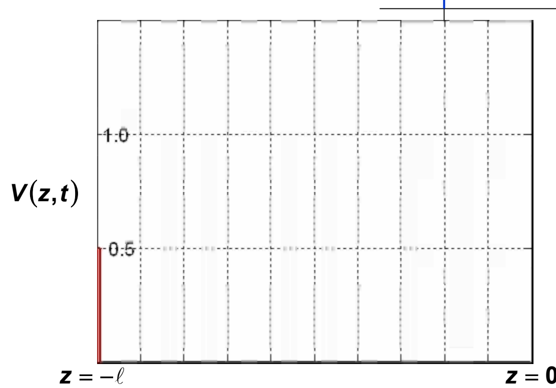


Step Voltage Source: Turn-On Transient – Capacitive Load - I

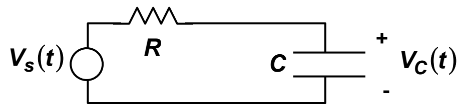


Source voltage is a step function: $V_s(t) = 1u(t)$



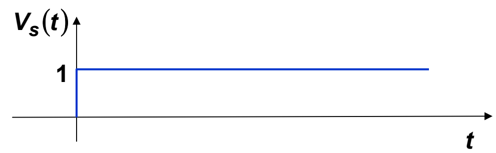


Step Voltage Source: Turn-On Transient in a RC Circuit



Source voltage is a step function:

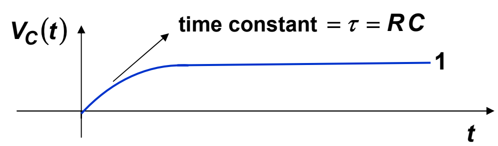
$$V_s(t) = 1u(t)$$



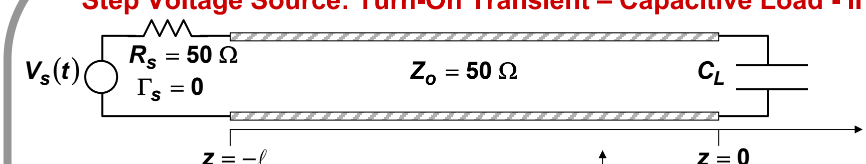
Solution for $V_C(t)$ is:

At short times the capacitor acts like a short
At long times the capacitor acts like an open

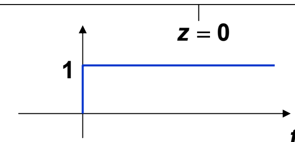
$$V_C(t) = \left(1 - e^{-\frac{t}{\tau}}\right)u(t)$$



Step Voltage Source: Turn-On Transient – Capacitive Load - II



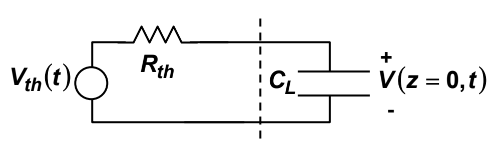
Source voltage is a step function: $V_s(t) = 1u(t)$



How does one solve this problem?

Transmission line is a linear system

So make a Thevenin equivalent circuit looking in from the load end

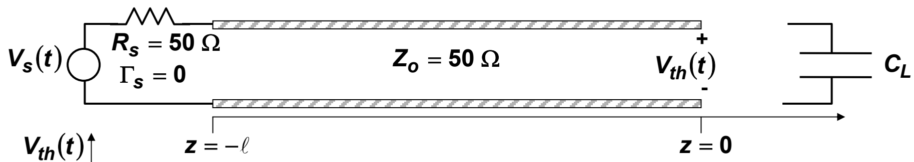


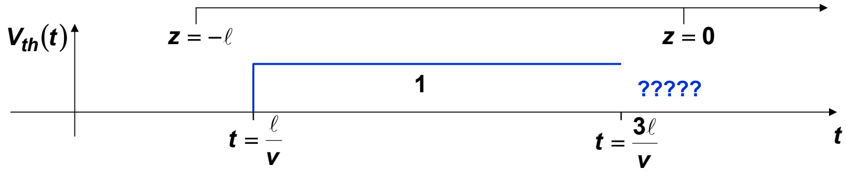
$R_{th} = Z_o$

This holds even if the source impedance were not matched to the line impedance (no such thing as impedance transformations in time domain)
This is because in transients situations it does not really matter what is on the other side of the line

Step Voltage Source: Turn-On Transient – Capacitive Load - II

To find $V_{th}(t)$ remove the load capacitor and look at the open circuit voltage:

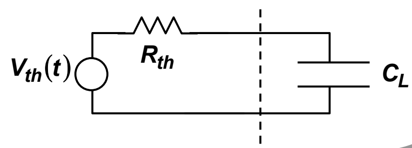




A forward voltage wave of 0.5 Volts from the source will reach the load at time $t = \frac{\ell}{v}$

A reflected voltage wave of 0.5 Volts will be generated at the same time $t = \frac{\ell}{v}$

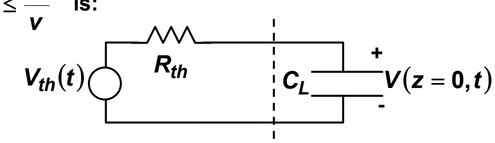
$\Rightarrow\ V_{th}(t) = 1$ for $\frac{\ell}{v} \le t \le \frac{3\ell}{v}$

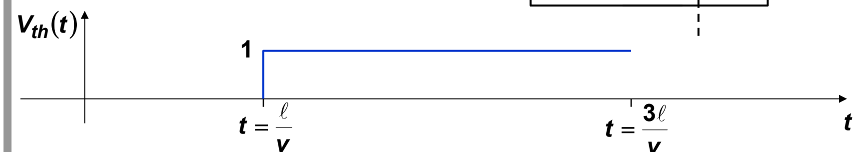


Step Voltage Source: Turn-On Transient – Capacitive Load - III

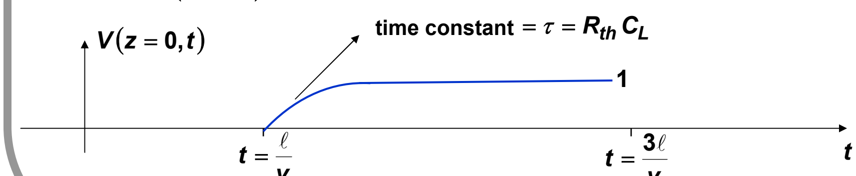
So the Thevenin circuit for times $\frac{\ell}{v} \le t \le \frac{3\ell}{v}$ is:

$R_{th} = Z_o$

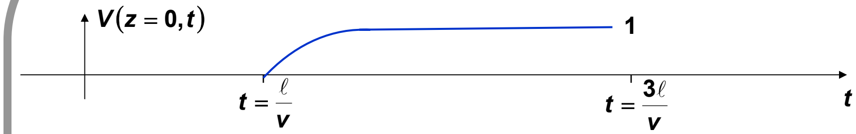




Solution for $V(z=0,t)$ is:

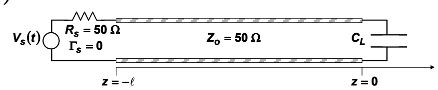


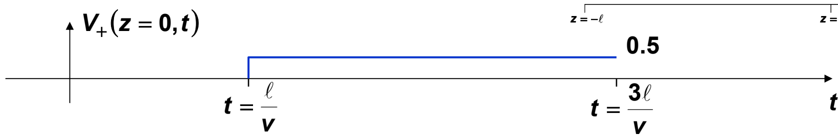
Step Voltage Source: Turn-On Transient – Capacitive Load - IV



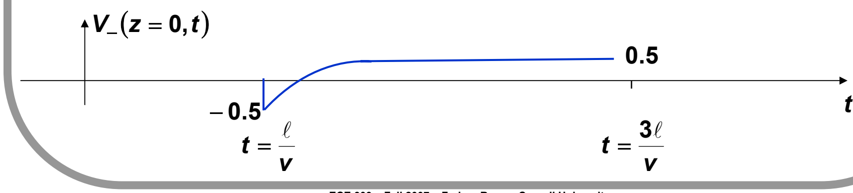
Since $V(z=0,t)=V_+(z=0,t)+V_-(z=0,t)$

And $V_+(z=0,t)$ is:

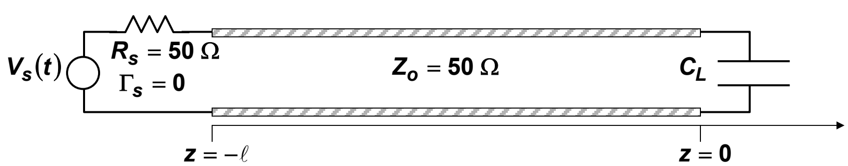


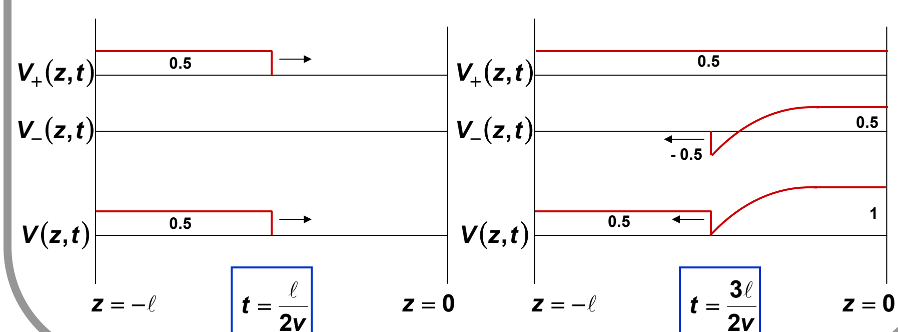


Therefore $V_-(z=0,t)$ must be:

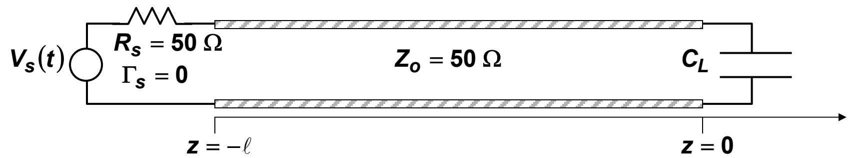


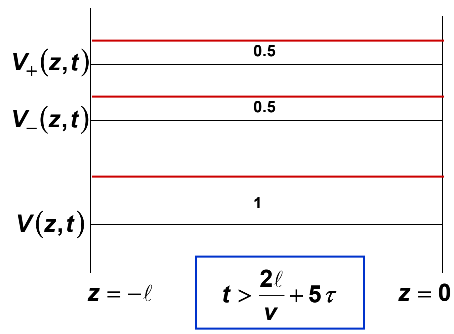
Step Voltage Source: Turn-On Transient – Capacitive Load - V





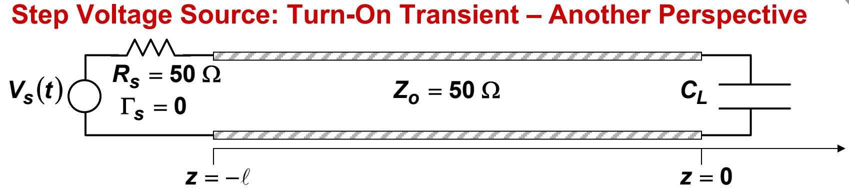
Step Voltage Source: Turn-On Transient – Capacitive Load - VI



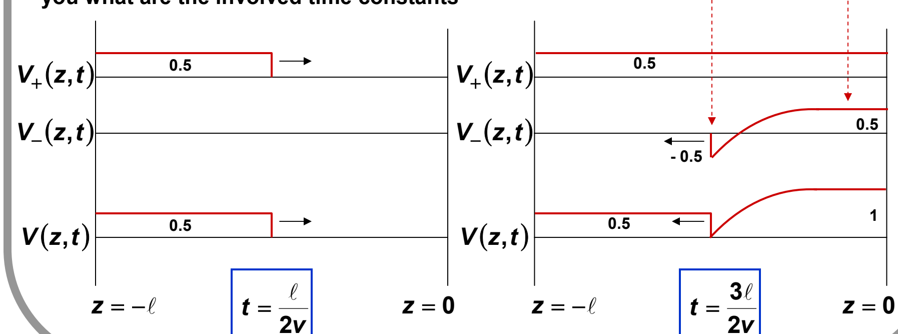


At sufficiently long times the capacitor is charged to 1 Volt

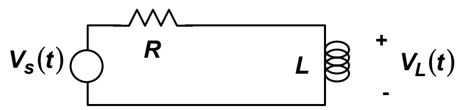
Step Voltage Source: Turn-On Transient – Another Perspective



At short times the capacitor acts like a short (Γ_L = -1)
At long times the capacitor acts like an open (Γ_L = +1)
CAREFUL: these kind of approximate arguments do not tell you what are the involved time constants

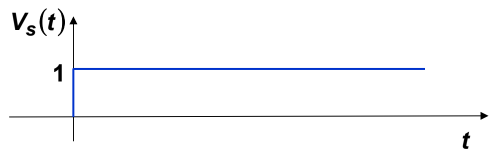


Step Voltage Source: Turn-On Transient in a LR Circuit



Source voltage is a step function:

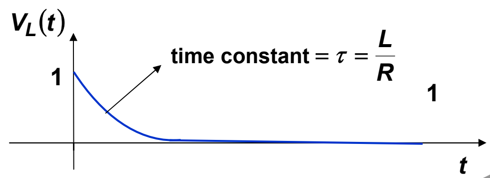
$$V_s(t) = 1u(t)$$



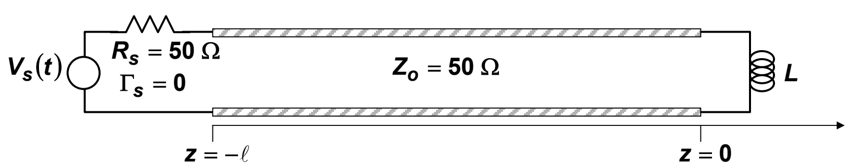
Solution for $V_L(t)$ is:

At short times the inductor acts like an open
At long times the inductor acts like a short

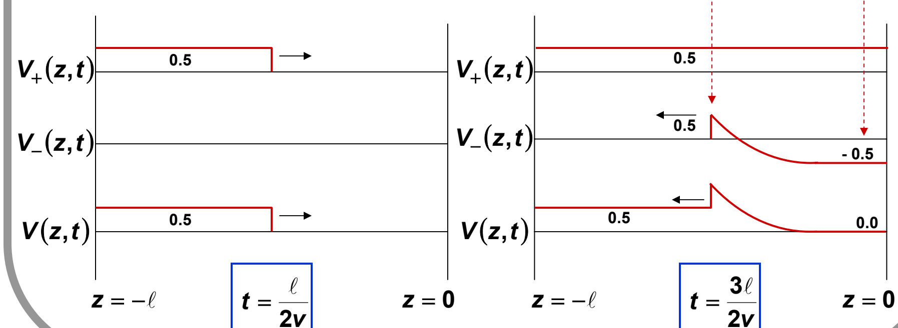
$$V_L(t) = e^{-\frac{t}{\tau}}\, u(t)$$



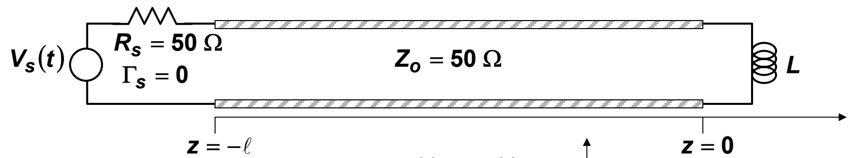
Step Voltage Source: Turn-On Transient –Inductive Load - I



Source voltage is a step function
At short times the inductor acts like an open (Γ_L = +1)
At long times the inductor acts like a short (Γ_L = -1)



Step Voltage Source: Turn-On Transient – Inductive Load - II



Source voltage is a step function: $V_s(t) = 1u(t)$