Lecture 23: Addition
Multilayer Structures – Non-Normal Incidence

In this lecture you will learn:

• Multilayer structures: non-normal incidence

Waves at Interfaces and Transmission Lines

\[ V(z)_{z<0} = V_{+1} e^{-j k_1 z} + V_{-1} e^{+j k_1 z} \]
\[ V(z)_{z>0} = V_{+2} e^{-j k_2 z} \]

Boundary conditions:

(1) \[ V_{+1} + V_{-1} = V_{+2} \]
(2) \[ \frac{V_{+1}}{Z_{01}} - \frac{V_{-1}}{Z_{01}} = \frac{V_{+2}}{Z_{02}} \]

\[ \Gamma = \frac{V_{-1}}{V_{+1}} = \frac{Z_{02}/Z_{01} - 1}{Z_{02}/Z_{01} + 1} \]

Boundary conditions:

(1) \[ E_{+1} + E_{-1} = E_{+2} \]
(2) \[ \frac{E_{+1}}{\eta_1} - \frac{E_{-1}}{\eta_1} = \frac{E_{+2}}{\eta_2} \]

\[ \Gamma = \frac{E_{-1}}{E_{+1}} = \frac{\eta_2/\eta_1 - 1}{\eta_2/\eta_1 + 1} \]
Waves at Interfaces: TE Wave - I

\[ E(x=0) = \hat{y} E_{+1} e^{-j(k_{1x} x + k_{1z} z)} + \hat{y} E_{-1} e^{-j(k_{1x} x - k_{1z} z)} \]

\[ E(x=\infty) = \hat{y} E_{+2} e^{-j(k_{2x} x + k_{2z} z)} \]

Boundary Conditions:

1. \( E_{+1} + E_{-1} = E_{+2} \)
2. \( \left[ \frac{E_{+1}}{\eta_1 \cos(\theta_1)} - \frac{E_{-1}}{\eta_1 \cos(\theta_1)} \right] = \frac{E_{+2}}{\eta_2 \cos(\theta_2)} \)

\[ T = \frac{E_{+2}}{E_{+1}} = \frac{2 \eta_2 \cos(\theta_2)}{\eta_1 \cos(\theta_1)} \left[ \frac{\eta_1 \cos(\theta_1)}{\eta_2 \cos(\theta_2)} + 1 \right] \]

\[ \Gamma = \frac{E_{-1}}{E_{+1}} = \frac{\eta_2 \cos(\theta_2)}{\eta_1 \cos(\theta_1)} \left[ \frac{\eta_1 \cos(\theta_1)}{\eta_2 \cos(\theta_2)} - 1 \right] \]

Waves at Interfaces: TE Wave - II

One may replace the above problem with a “dummy” normal incidence problem:

\[ m \cos(\theta) \]

\[ E_{+1} \]

\[ k = k_{1z} z \]

\[ E_{+2} \]

\[ k = k_{2z} z \]

And then calculate the reflection and transmission coefficients for the E-field:

\[ \Gamma = \frac{E_{-1}}{E_{+1}} = \frac{\eta_2 \cos(\theta_2)}{\eta_1 \cos(\theta_1)} \left[ \frac{\eta_1 \cos(\theta_1)}{\eta_2 \cos(\theta_2)} - 1 \right] \]

\[ T = \frac{E_{+2}}{E_{+1}} = \frac{2 \eta_2 \cos(\theta_2)}{\eta_1 \cos(\theta_1)} \left[ \frac{\eta_1 \cos(\theta_1)}{\eta_2 \cos(\theta_2)} + 1 \right] \]
Waves at Interfaces: TE Wave - III

In the “dummy” normal incidence problem:

- The wavevectors are taken to be the z-component of the actual wavevectors in each medium.
- The impedances in each medium are taken to be the actual impedances divided by the cosines of the angles of propagation w.r.t. the z-axis.

\[ \Gamma = \frac{E_{s1}}{E_{s2}} = \frac{\eta_2 / \cos(\theta_2) - 1}{\eta_1 / \cos(\theta_1) + 1} \]

\[ T = \frac{E_{s2}}{E_{s1}} = \frac{2 \eta_2 / \cos(\theta_2)}{\eta_1 / \cos(\theta_1) + 1} \]

Warning: The “dummy” normal incidence problem is only meant to be used to calculate reflection and transmission coefficients – don’t use this framework for anything else.

Tri-layer Structure: TE Wave - I

How does one solve a problem like this?

\[ \Gamma = \frac{E_{s1}}{E_{s2}} = ? \]
Tri-layer Structure: TE Wave - II

Replace the above problem with a "dummy" normal incidence problem:

And then use the usual methods to solve it (impedance transformations, Smith chart, etc)

Waves at Interfaces: TM Wave - I

Work with the E-field component parallel to the media interface (i.e. the x-component)

Boundary Conditions:

(1) \[ E_{+1x} + E_{-1x} = E_{+2x} \]

(2) \[
\begin{bmatrix}
E_{+1x} \\
E_{-1x}
\end{bmatrix}
\begin{bmatrix}
\eta_1 \cos(\theta_1) \\
\eta_1 \cos(\theta_1)
\end{bmatrix} =
\begin{bmatrix}
E_{+2x} \\
E_{-2x}
\end{bmatrix}
\]

\[
\begin{pmatrix}
\eta_2 \cos(\theta_2) - 1 \\
\eta_1 \cos(\theta_1)
\end{pmatrix}
\begin{pmatrix}
\eta_1 \cos(\theta_1) \\
\eta_1 \cos(\theta_1)
\end{pmatrix}
= \begin{pmatrix}
\eta_2 \cos(\theta_2) + 1 \\
\eta_1 \cos(\theta_1)
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_{+1x} \\
E_{-1x}
\end{pmatrix}
\begin{pmatrix}
\eta_1 \cos(\theta_1) \\
\eta_1 \cos(\theta_1)
\end{pmatrix} =
\begin{pmatrix}
\eta_2 \cos(\theta_2) \\
\eta_1 \cos(\theta_1)
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_{+2x} \\
E_{-2x}
\end{pmatrix}
\begin{pmatrix}
\eta_1 \cos(\theta_1) \\
\eta_1 \cos(\theta_1)
\end{pmatrix} = \begin{pmatrix}
\eta_2 \cos(\theta_2) + 1 \\
\eta_1 \cos(\theta_1)
\end{pmatrix}
\]
One may replace the above problem with a “dummy” normal incidence problem:

And then calculate the reflection and transmission coefficients for the x-component of the E-field:

\[
\Gamma = \frac{E_{-1x}}{E_{+1x}} = \frac{\eta_2 \cos(\theta_2)}{\eta_1 \cos(\theta_1)} - 1
\]

\[
T = \frac{E_{+2x}}{E_{+1x}} = \frac{2 \eta_2 \cos(\theta_2)}{\eta_1 \cos(\theta_1)} + 1
\]

Work with the E-field component parallel to the media interface (i.e. the x-component)

\[
E_x(r) |_{z<-l} = E_{+1x} e^{-j(k_{1x} x + k_{1z} (z+l))} + E_{-1x} e^{-j(k_{1x} x - k_{1z} (z+l))}
\]

\[
E_x(r) |_{-l < z < 0} = E_{+2x} e^{-j(k_{2x} x + k_{2z} z)} + E_{-2x} e^{-j(k_{2x} x - k_{2z} z)}
\]

\[
E_x(r) |_{z>0} = E_{+3x} e^{-j(k_{3x} x + k_{3z} z)}
\]

How does one solve a problem like this?

\[
\Gamma = \frac{E_{-1x}}{E_{+1x}} = ?
\]
Tri-layer Structure: TM Wave - II

Replace the above problem with a “dummy” normal incidence problem:

And then use the usual methods to solve it (impedance transformations, Smith chart, etc)