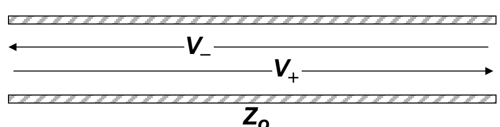
Lecture 21

Transmission Lines: RF and Microwave Circuits

In this lecture you will learn:

- More about transmission lines
- Impedance transformation in transmission lines
- Transmission line circuits and systems

Transmission Lines: A Review



Voltage at any point on the line can be written as:

$$V(z) = V_+ \, e^{-j k z} + V_- \, e^{+j k z}$$

Current at any point on the line can be written as:

$$I(z) = \frac{V_+}{Z_o} e^{-j k z} - \frac{V_-}{Z_o} e^{+j k z}$$

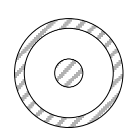
The characteristic impedance of a transmission line is:

$$Z_o = \sqrt{\frac{L}{C}}$$

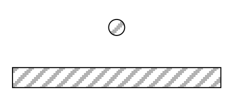
The dispersion relation for a transmission line is:

$$k = \omega \sqrt{LC}$$

Some common examples of transmission lines



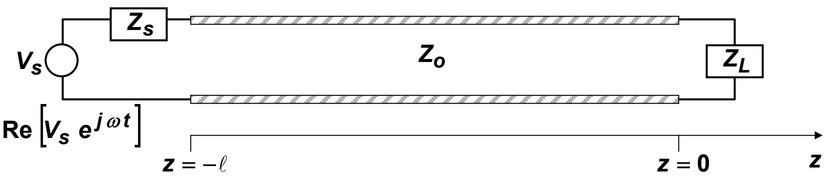
Co-axial line



Wire on a ground plane

Transmission Line Circuits

Consider a transmission line connected as shown below:



$$V_s(t) = \mathrm{Re}\left[V_s\, e^{j\omega t}\right]$$

Z_o = Transmission line impedance
Z_L = Load impedance
Z_s = Source impedance

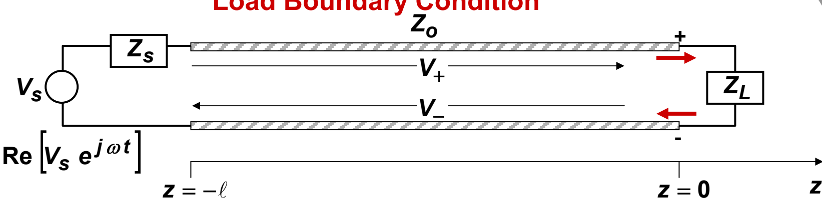
In general, voltage on a transmission line is a superposition of forward and backward going waves:

$$V(z) = V_+\, e^{-jkz} + V_-\, e^{+jkz}$$

The corresponding current is also a superposition of forward and backward going waves:

$$I(z) = \frac{V_+}{Z_o} e^{-jkz} - \frac{V_-}{Z_o} e^{+jkz}$$

Load Boundary Condition



$$V_s(t) = \mathrm{Re}\left[V_s\, e^{j\omega t}\right]$$

$$V(z) = V_+\, e^{-jkz} + V_-\, e^{+jkz} \qquad I(z) = \frac{V_+}{Z_o} e^{-jkz} - \frac{V_-}{Z_o} e^{+jkz}$$

Boundary condition:

At $z = 0$ the ratio of the total voltage to the total current must equal the load impedance:

$$\frac{V(z=0)}{I(z=0)} = \frac{V_+ + V_-}{V_+/Z_o - V_-/Z_o} = Z_L$$

$$\Rightarrow \frac{V_-}{V_+} = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} \longrightarrow$$

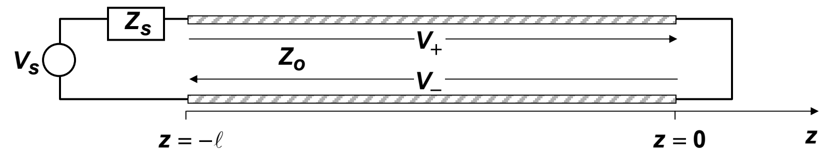
This gives us the backward going wave amplitude in terms of the forward going wave amplitude

Define a load reflection coefficient Γ_L as:

$$\Gamma_L = \frac{V_-}{V_+} = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1}$$

Load Reflections

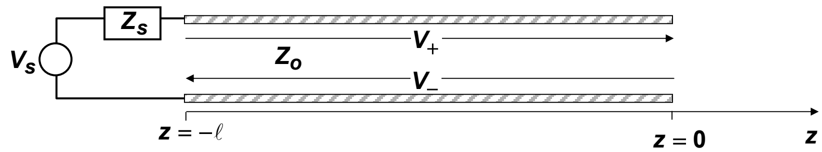
Suppose Z_L = 0 (short):



$$\Gamma_L = \frac{V_-}{V_+} = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} = -1 \quad \Rightarrow \quad V_- = -V_+$$

$$V(z=0) = V_+ + V_- = 0 \quad \text{and} \quad I(z=0) = \frac{V_+}{Z_o} - \frac{V_-}{Z_o} = 2\frac{V_+}{Z_o}$$

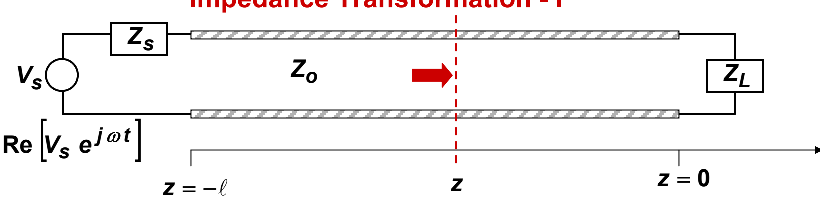
Suppose Z_L = ∞ (open):



$$\Gamma_L = \frac{V_-}{V_+} = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} = +1 \quad \Rightarrow \quad V_- = V_+$$

$$V(z=0) = V_+ + V_- = 2V_+ \quad \text{and} \quad I(z=0) = \frac{V_+}{Z_o} - \frac{V_-}{Z_o} = 0$$

Impedance Transformation - I



$$V_s(t) = \text{Re}\left[V_s\, e^{j\omega t}\right]$$

$$V(z) = V_+\left(e^{-jkz} + \Gamma_L\, e^{+jkz}\right) \qquad I(z) = \frac{V_+}{Z_o}\left(e^{-jkz} - \Gamma_L\, e^{+jkz}\right)$$

Question: What is the impedance *Z*(*z*) looking towards the load at the location *z* in the transmission line?

$$Z(z) = \frac{V(z)}{I(z)} = Z_o\,\frac{1+\Gamma_L\, e^{2jkz}}{1-\Gamma_L\, e^{2jkz}} \longrightarrow \left\{ Z(z) = R(z) + j\,X(z) \right.$$

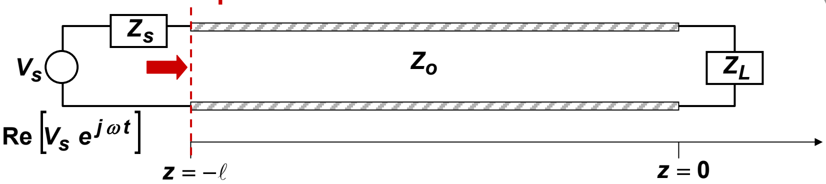
$R(z)$ → **resistance**

$X(z)$ → **reactance**

Check: What is the impedance *Z*(*z*=0)?

$$Z(z=0) = Z_o\left.\frac{1+\Gamma_L\, e^{2jkz}}{1-\Gamma_L\, e^{2jkz}}\right|_{z=0} = Z_o\,\frac{1+\Gamma_L}{1-\Gamma_L} = Z_L$$

Impedance Transformation - II



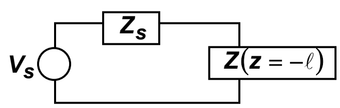
$$V_s(t) = \text{Re}\left[V_s\, e^{j\omega t}\right]$$

$$V(z) = V_+\left(e^{-jkz} + \Gamma_L\, e^{+jkz}\right) \qquad I(z) = \frac{V_+}{Z_o}\left(e^{-jkz} - \Gamma_L\, e^{+jkz}\right)$$

Question: What is the impedance $Z(z=-\ell)$ looking towards the load at the location $z=-\ell$ in the transmission line

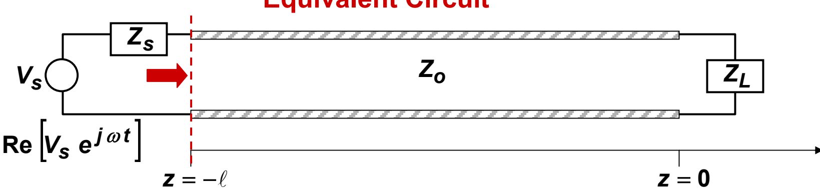
$$Z(z=-\ell) = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_o \left.\frac{1+\Gamma_L\, e^{2jkz}}{1-\Gamma_L\, e^{2jkz}}\right|_{z=-\ell} = Z_o \frac{1+\Gamma_L\, e^{-2jk\ell}}{1-\Gamma_L\, e^{-2jk\ell}}$$

Knowing the impedance looking into the line at $z=-\ell$ we can use the following equivalent circuit:

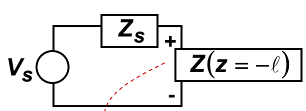


The load impedance Z_L has been transformed by the transmission line into the impedance $Z(z=-\ell)$ at the other end

Equivalent Circuit



$$V_s(t) = \text{Re}\left[V_s\, e^{j\omega t}\right]$$



$$\begin{cases} V(z) = V_+\left(e^{-jkz} + \Gamma_L\, e^{+jkz}\right) \\ I(z) = \dfrac{V_+}{Z_o}\left(e^{-jkz} - \Gamma_L\, e^{+jkz}\right) \end{cases}$$

The voltage across the impedance $Z(z=-\ell)$ in the above circuit is:

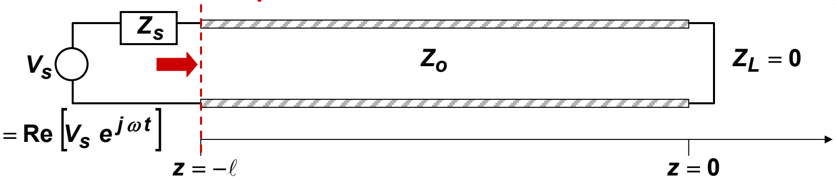
$$V_s \frac{Z(z=-\ell)}{Z_s + Z(z=-\ell)}$$

So on the transmission line we must have:

$$V(z=-\ell) = V_+\left(e^{jk\ell} + \Gamma_L\, e^{-jk\ell}\right) = V_s \frac{Z(z=-\ell)}{Z_s + Z(z=-\ell)}$$

V_+ can be found from the above equation

Example – Short Circuit Load - I



$$V_s(t) = \text{Re}\left[V_s\, e^{j\omega t}\right]$$

Z_L = 0 implies:

$$\Gamma_L = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} = -1 \qquad \text{and} \qquad Z(z=-\ell) = Z_o\,\frac{1+\Gamma_L\, e^{-2jk\ell}}{1-\Gamma_L\, e^{-2jk\ell}} = Z_o\; j\tan(k\,\ell)$$

Suppose $k\ell$ << 1:

$$k\ell \ll 1 \;\Rightarrow\; \frac{2\pi}{\lambda}\ell \ll 1 \;\Rightarrow\; \ell \ll \frac{\lambda}{2\pi}$$

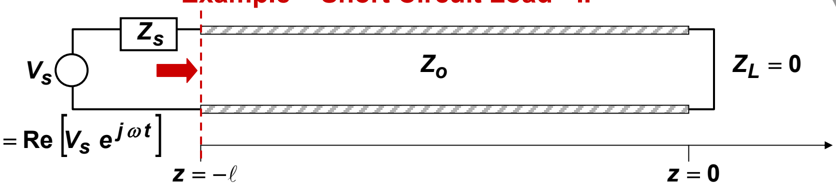
$$k\ell \ll 1 \;\Rightarrow\; \frac{\omega}{v}\ell \ll 1 \;\Rightarrow\; \omega \ll \frac{v}{\ell}$$

Then:

$$Z(z=-\ell) = Z_o\; j\tan(k\,\ell) \approx j\, Z_o\, k\ell = j\sqrt{\frac{L}{C}}\,\omega\sqrt{LC}\;\ell = j\,\omega\,(L\ell)$$

- Impedance seen at the source end is inductive
- The transmission line appears like one big inductor

Example – Short Circuit Load - II

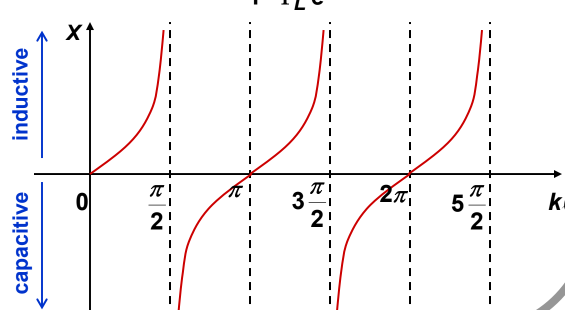


$$V_s(t) = \text{Re}\left[V_s\, e^{j\omega t}\right]$$

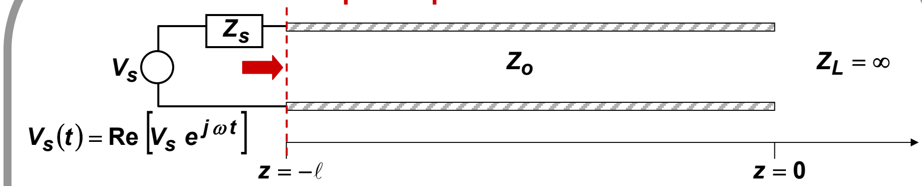
Z_L = 0 implies:

$$\Gamma_L = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} = -1 \qquad \text{and} \qquad Z(z=-\ell) = Z_o\,\frac{1+\Gamma_L\, e^{-2jk\ell}}{1-\Gamma_L\, e^{-2jk\ell}} = Z_o\; j\tan(k\,\ell)$$

$$Z(z=-\ell) = j\, Z_o\, \tan(k\,\ell) = j\,X$$



Example – Open Circuit Load - I



$Z_L = \infty$ implies:

$$\Gamma_L = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} = +1 \quad \text{and} \quad Z(z=-\ell) = Z_o \frac{1+\Gamma_L e^{-2jk\ell}}{1-\Gamma_L e^{-2jk\ell}} = -Z_o\, j\cot(k\ell)$$

Suppose $k\ell << 1$:

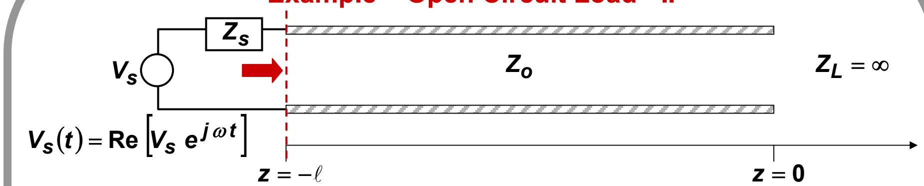
$$k\ell << 1 \;\Rightarrow\; \frac{2\pi}{\lambda}\ell << 1 \;\Rightarrow\; \ell << \frac{\lambda}{2\pi}$$

$$k\ell << 1 \;\Rightarrow\; \frac{\omega}{v}\ell << 1 \;\Rightarrow\; \omega << \frac{v}{\ell}$$

Then: $$Z(z=-\ell) = -Z_o\, j\cot(k\ell) \approx -j\frac{Z_o}{k\ell} = -j\sqrt{\frac{L}{C}}\frac{1}{\omega\sqrt{LC}\,\ell} = \frac{1}{j\omega(C\ell)}$$

- Impedance seen at the source end is capacitive
- The transmission line appears like one big capacitor

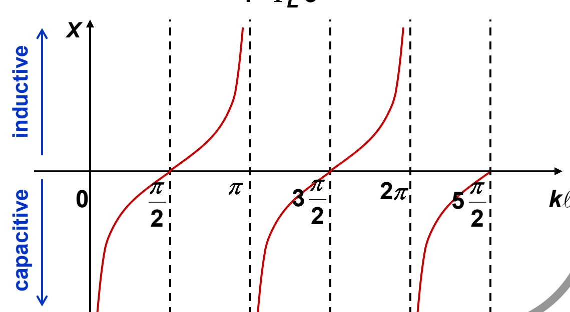
Example – Open Circuit Load - II



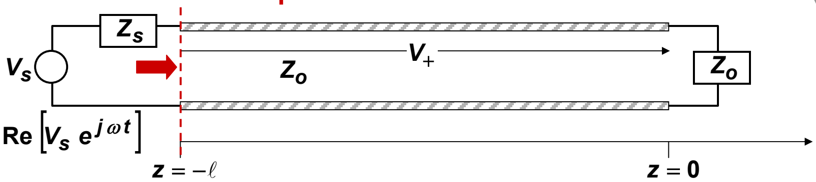
$Z_L = \infty$ implies:

$$\Gamma_L = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} = +1 \quad \text{and} \quad Z(z=-\ell) = Z_o \frac{1+\Gamma_L e^{-2jk\ell}}{1-\Gamma_L e^{-2jk\ell}} = -Z_o\, j\cot(k\ell)$$

$$Z(z=-\ell) = -j\,Z_o\cot(k\ell) = jX$$



Example – Matched Load



$$V_s(t) = \text{Re}\left[V_s\, e^{j\omega t}\right]$$

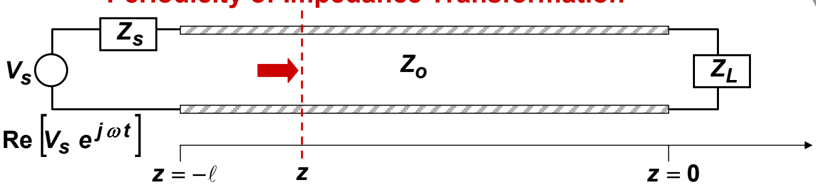
$Z_L = Z_o$ implies:

$$\Gamma_L = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} = 0 \qquad \text{and} \qquad Z(z=-\ell) = Z_o \frac{1+\Gamma_L e^{-2jk\ell}}{1-\Gamma_L e^{-2jk\ell}} = Z_o$$

For matched loads:

- The load reflection coefficient Γ_L is zero
- There is no reflected wave generated at the load end (i.e. $V_- = 0$)
- The impedance seen at the source end is Z_o irrespective of the length of the transmission line

Periodicity of Impedance Transformation



$$V_s(t) = \text{Re}\left[V_s\, e^{j\omega t}\right]$$

$$\Gamma_L = \frac{Z_L/Z_o - 1}{Z_L/Z_o + 1} \qquad \text{and} \qquad Z(z) = Z_o \frac{1+\Gamma_L e^{2jkz}}{1-\Gamma_L e^{2jkz}}$$

For locations z such that $2kz = 2m\pi$ (m is an integer) then:

$$Z(z) = Z_o \frac{1+\Gamma_L e^{2jkz}}{1-\Gamma_L e^{2jkz}} = Z_o \frac{1+\Gamma_L}{1-\Gamma_L} = Z_L \longrightarrow$$

Impedance seen at locations $z = m\pi/k$ is also Z_L

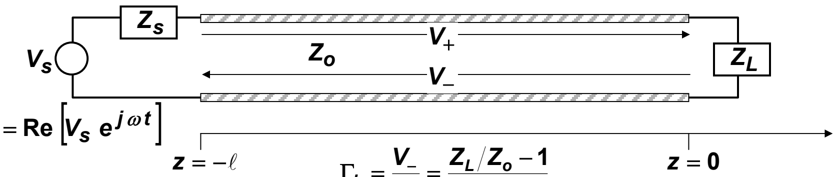
Note that: $2kz = 2m\pi$ but $k = \dfrac{2\pi}{\lambda} \Rightarrow z = m\dfrac{\lambda}{2}$

Impedance seen at locations z that are integer multiples of half-wavelength from the load end is the load impedance Z_L

More generally: $$Z\left(z \pm m\frac{\lambda}{2}\right) = Z(z)$$

Impedance is periodic with period equal to half-wavelength

Standing Wave Ratio



$$V_s(t)=\text{Re}\left[V_s\, e^{j\omega t}\right]$$

$$\Gamma_L=\frac{V_-}{V_+}=\frac{Z_L/Z_o-1}{Z_L/Z_o+1}$$

$$V(z)=V_+\,e^{-jkz}+V_-\,e^{+jkz}=V_+\left(e^{-jkz}+\Gamma_L\,e^{+jkz}\right)\longrightarrow\left\{\Gamma_L=|\Gamma_L|\,e^{j\phi}\right.$$

$$\Rightarrow\quad |V(z)|^2=|V_+|^2\left[1+|\Gamma_L|^2+2\,|\Gamma_L|\cos(2kz+\phi)\right]$$

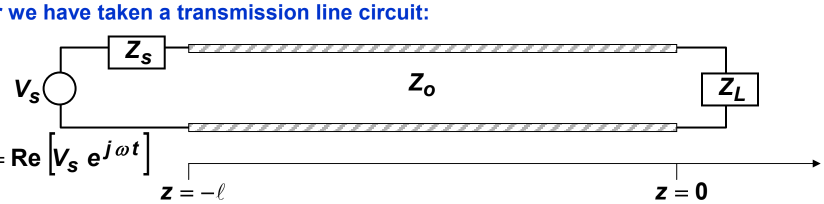
$$\Rightarrow\quad |V(z)|=|V_+|\sqrt{1+|\Gamma_L|^2+2\,|\Gamma_L|\cos(2kz+\phi)}$$

The interference of forward and backward going waves leads to standing-wave-like behavior

$$\text{Standing Wave Ratio}\;=SWR=\frac{|V(z)|_{\max}}{|V(z)|_{\min}}=\frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$

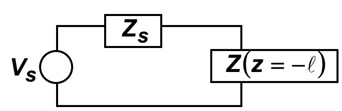
Thevenin Equivalents of the Source - I

So far we have taken a transmission line circuit:

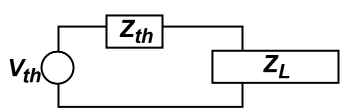


$$V_s(t)=\text{Re}\left[V_s\, e^{j\omega t}\right]$$

And reduced it to the equivalent circuit:



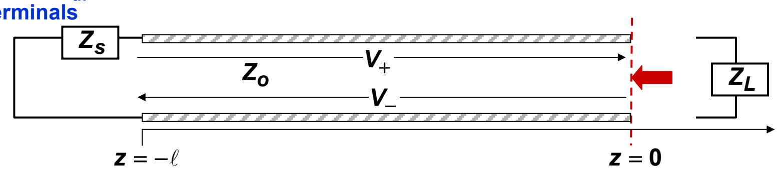
Now we want to reduce it to the equivalent circuit:



i.e. we want to produce the Thevenin equivalent of the "source + transmission line" as seen by the load – How do we do that?

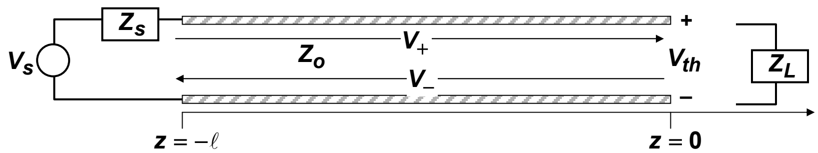
Thevenin Equivalents of the Source - II

STEP 1: To find Z_{th} short the voltage source and find the impedance looking in from the load terminals



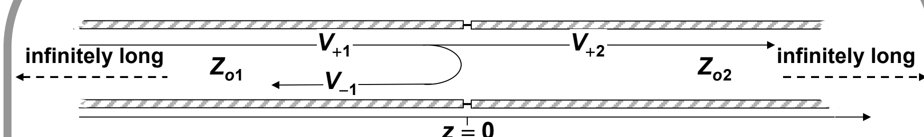
$$\Gamma_s = \frac{V_+}{V_-} = \frac{Z_s/Z_o - 1}{Z_s/Z_o + 1} \quad \text{and} \quad Z_{th} = Z_o \frac{1 + \Gamma_s e^{-2jk\ell}}{1 - \Gamma_s e^{-2jk\ell}}$$

STEP 2: To find V_{th} remove the load and find the voltage at the load terminals



$$V_{th} = 2V_+ = \frac{V_s}{\cos(k\ell)} \, \frac{-jZ_o \cot(k\ell)}{Z_s - jZ_o \cot(k\ell)}$$

Unmatched Infinite Transmission Lines



Consider the situation above where we have two unmatched transmission lines connected together

$$V_1(z) = V_{+1} e^{-jk_1 z} + V_{-1} e^{+jk_1 z}$$

$$I_1(z) = \frac{V_{+1}}{Z_{o1}} e^{-jk_1 z} - \frac{V_{-1}}{Z_{o1}} e^{+jk_1 z}$$

$$V_2(z) = V_{+2} e^{-jk_2 z}$$

$$I_2(z) = \frac{V_{+2}}{Z_{o2}} e^{-jk_2 z}$$

Boundary Conditions

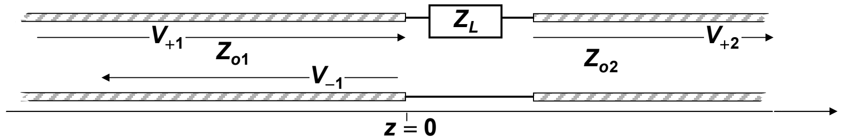
(1) At z=0 the voltage on both the transmission lines must be the same
(2) At z=0 the current on both the transmission lines must be the same

$$(1) \Rightarrow V_{+1} + V_{-1} = V_{+2} \quad \text{and} \quad (2) \Rightarrow \frac{V_{+1}}{Z_{o1}} - \frac{V_{-1}}{Z_{o1}} = \frac{V_{+2}}{Z_{o2}}$$

$$\Gamma = \frac{V_{-1}}{V_{+1}} = \frac{Z_{o2}/Z_{o1} - 1}{Z_{o2}/Z_{o1} + 1} \qquad T = \frac{V_{+2}}{V_{+1}} = \frac{2 Z_{o2}/Z_{o1}}{Z_{o2}/Z_{o1} + 1}$$

How does one avoid reflections at the junction of the two transmission lines?

Power Splitting in Microwave Circuits



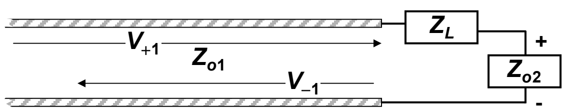
Input wave: $V_i(z) = V_{+1}\, e^{-j k_1 z}$

Transmitted wave: $V_t(z) = V_{+2}\, e^{-j k_2 z}$

Reflected wave: $V_r(z) = V_{-1}\, e^{+j k_1 z}$

Goal: Need to find V_{+2} and V_{-1} in terms of V_{+1}

STEP 1: Cast the circuit in the following equivalent form and find V_{-1}



$$\Gamma = \frac{V_{-1}}{V_{+1}} = \frac{(Z_L + Z_{o2})/Z_{o1} - 1}{(Z_L + Z_{o2})/Z_{o1} + 1}$$

STEP 2: Voltage V_{+2} is the same as the voltage across the impedance Z_{o2} in the equivalent circuit

$$V_{+2} = \left[V_i(z=0) + V_r(z=0)\right] \frac{Z_{o2}}{Z_L + Z_{o2}} = V_{+1}\,(1+\Gamma)\,\frac{Z_{o2}}{Z_L + Z_{o2}}$$