Lecture 20

Transmission Lines: The Basics

In this lecture you will learn:

- Transmission lines
- Different types of transmission line structures
- Transmission line equations
- Power flow in transmission lines
- Appendix

Guided Waves

- So far in the course you have been dealing with waves that propagated in infinite size media
- For many applications it is desirable to have electromagnetic energy be guided in much the same way as water flow is guided by having it flow in pipes
- Transmission lines are the simplest structures that guide electromagnetic waves

Transmission line:

In this course a transmission line would be any two arbitrary shaped metal conductors that are very long and uniform in at least one dimension

A transmission line made up of two metal conductors that are very long and uniform in the z-direction
Common Transmission Line Structures - I

Co-axial Cable

\[
C = \frac{2\pi \varepsilon}{\log\left(\frac{b}{a}\right)}
\]

\[
L = \frac{\mu_0}{2\pi} \log\left(\frac{b}{a}\right)
\]

\[
LC = \mu_0 \varepsilon
\]

Parallel-Plate Transmission Line

\[
C = \varepsilon \frac{W}{d}
\]

\[
L = \mu_0 \frac{d}{W}
\]

\[
LC = \mu_0 \varepsilon
\]

Capacitance and Inductance per unit length for each structure are shown.

Common Transmission Line Structures - II

Parallel Metal Wires

For \( d \gg a \):

\[
C = \frac{\pi \varepsilon_0}{\log\left(\frac{d}{a}\right)}
\]

\[
L = \frac{\mu_0}{\pi} \log\left(\frac{d}{a}\right)
\]

\[
LC = \mu_0 \varepsilon_0
\]

Metal Wire Over a Ground Plane

For \( d \gg a \):

\[
C = \frac{2\pi \varepsilon_0}{\log\left(\frac{d}{a}\right)}
\]

\[
L = \frac{\mu_0}{2\pi} \log\left(\frac{d}{a}\right)
\]

\[
LC = \mu_0 \varepsilon_0
\]

Capacitance and Inductance per unit length for each structure are shown.
Transmission Line Voltages

- Suppose the potential difference between the two conductors of a transmission line at location $z$ is $V(z)$ then E-field line integral in a plane parallel to the $x$-$y$ plane at the location $z$ is independent of the contour taken, and is related to the potential difference $V(z)$ as:

$$V(z) = \int_{c_1}^{c_2} \vec{E} \cdot d\vec{s} = \int_{c_1}^{c_2} \vec{E} \cdot ds$$

- The charge per unit length $Q(z)$ on the transmission line at location $z$ is:

$$Q(z) = CV(z)$$

Two points with different $z$-values can have different potentials:

$$V(z_1) = \int_{c_1}^{c_2} \vec{E} \cdot ds \neq \int_{c_1}^{c_2} \vec{E} \cdot ds = V(z_2)$$

The conductors are no longer equi-potentials

Transmission Line Currents

- Suppose the total current in the upper conductor in the $+z$-direction at location $z$ is $I(z)$ and in the lower conductor is $-I(z)$

- The H-field flux per unit length $\lambda(z)$ enclosed between the two conductors at the location $z$ is related to the current $I(z)$ as:

$$\lambda(z) = LI(z)$$

Two points with different $z$-values can have different currents

$$I(z_1) \neq I(z_2)$$
Faraday's Law

Use Faraday's law for the contour C:

\[
\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int \mu_0 \mathbf{H} \cdot d\mathbf{a}
\]

\[
\Rightarrow V(z+\Delta z,t) - V(z,t) = \frac{\partial \lambda(z,t)}{\partial t} \frac{\Delta z}{\Delta z} \Rightarrow \lambda(z,t) = L I(z,t)
\]

\[
\Rightarrow \frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = \frac{\partial L}{\partial t} I(z,t) \Rightarrow \frac{\partial V(z,t)}{\partial z} = -L \frac{\partial I(z,t)}{\partial t}
\]

\[Q(z,t) = CV(z,t)\]

Current-Charge Continuity Equation

Use the principle of conservation of charge (current-charge continuity equation):

If current is varying in space, there must be charge either piling up or piling down somewhere

\[
I(z,t) - I(z+\Delta z,t) = \frac{\partial Q(z,t)}{\partial t} \frac{\Delta z}{\Delta z} \Rightarrow Q(z,t) = CV(z,t)
\]

\[
\Rightarrow I(z+\Delta z,t) - I(z,t) = -C \frac{\partial V(z,t)}{\partial t} \Rightarrow \frac{\partial I(z,t)}{\partial z} = -C \frac{\partial V(z,t)}{\partial t}
\]

\[Q(z,t) = CV(z,t)\]
Transmission Line Equations

The following two equations describe the propagation of guided electromagnetic waves on transmission lines (also called the Telegrapher’s Equations):

\[
\frac{\partial V(z,t)}{\partial z} = -L \frac{\partial I(z,t)}{\partial t} \quad (1)
\]

\[
\frac{\partial I(z,t)}{\partial z} = -C \frac{\partial V(z,t)}{\partial t} \quad (2)
\]

**Wave equations:**

\[
(1) \Rightarrow \frac{\partial^2 V(z,t)}{\partial z^2} = -L \frac{\partial^2 I(z,t)}{\partial z \partial t} \quad (3)
\]

\[
(2) \Rightarrow \frac{\partial^2 I(z,t)}{\partial z \partial t} = -C \frac{\partial^2 V(z,t)}{\partial t^2} \quad (4)
\]

\[
(3) \text{ and } (4) \Rightarrow \frac{\partial^2 V(z,t)}{\partial z^2} = LC \frac{\partial^2 V(z,t)}{\partial t^2} \quad \text{Equation of a wave traveling with a velocity } v = \frac{1}{\sqrt{LC}}
\]

A similar equation can be derived for the current:

\[
\frac{\partial^2 I(z,t)}{\partial z^2} = LC \frac{\partial^2 I(z,t)}{\partial t^2}
\]

Nature of Guided Waves in Transmission Lines - I

\[
\frac{\partial^2 V(z,t)}{\partial z^2} = LC \frac{\partial^2 V(z,t)}{\partial t^2} \quad \frac{\partial^2 I(z,t)}{\partial z^2} = LC \frac{\partial^2 I(z,t)}{\partial t^2}
\]

The guided wave consists of E-fields and H-fields together with charges and currents on the conductors that all move together in sync with a velocity given by:

\[
v = \frac{1}{\sqrt{LC}}
\]

**y**

*The charges satisfy the boundary conditions for the E-fields*
Nature of Guided Waves in Transmission Lines - II

\[ \frac{\partial^2 V(z,t)}{\partial z^2} + \frac{1}{LC} \frac{\partial^2 V(z,t)}{\partial t^2} = 0 \]

\[ \frac{\partial^2 I(z,t)}{\partial z^2} + \frac{1}{LC} \frac{\partial^2 I(z,t)}{\partial t^2} = 0 \]

The guided wave consists of E-fields and H-fields together with charges and currents on the conductors that all move together in sync with a velocity given by:

\[ v = \frac{1}{\sqrt{LC}} \]

- The charges satisfy the boundary conditions for the E-fields
- The currents satisfy the boundary conditions for the H-fields
- The wave is called a TEM wave since both the E-field and H-field point in a direction transverse to the direction of propagation

E-Fields and H-fields for Common Transmission Lines

Notice that at each point \( \vec{E}(r,t) \times \vec{H}(r,t) \) points in the +z-direction – indicating energy flow in the +z-direction
Notice that at each point $\vec{E}(\vec{r}, t) \times \vec{H}(\vec{r}, t)$ points in the $+z$-direction – indicating energy flow in the $+z$-direction.

Current and Voltage Phasors

Convert to phasors:

\[
V(z, t) = \text{Re} \left[ V(z) e^{j\omega t} \right] \\
l(z, t) = \text{Re} \left[ l(z) e^{j\omega t} \right]
\]

Transmission line equations in phasor notation:

\[
\frac{\partial V(z, t)}{\partial z} = -L \frac{\partial l(z, t)}{\partial t} \quad \frac{\partial V(z)}{\partial z} = -j \omega L l(z) \\
\frac{\partial l(z, t)}{\partial z} = -C \frac{\partial V(z, t)}{\partial t} \quad \frac{\partial l(z)}{\partial z} = -j \omega C V(z)
\]

Wave equations in phasor notation:

\[
\frac{\partial^2 V(z, t)}{\partial z^2} = \frac{1}{LC} \frac{\partial^2 V(z, t)}{\partial t^2} \quad \frac{\partial^2 V(z)}{\partial z^2} = -\omega^2 LC V(z) \\
\frac{\partial^2 l(z, t)}{\partial z^2} = \frac{1}{LC} \frac{\partial^2 l(z, t)}{\partial t^2} \quad \frac{\partial^2 l(z)}{\partial z^2} = -\omega^2 LC l(z)
\]
Solutions of Transmission Line Equations

Start with the complex wave equation:
\[ \frac{\partial^2 V(z)}{\partial z^2} = -\omega^2 LC V(z) \]

Assume a solution of the form of a traveling wave:
\[ V(z) = V_+ e^{-j k z} \]

A wave traveling in the +z-direction

Substitute in the complex wave equation:
\[ \frac{\partial^2 V(z)}{\partial z^2} = -\omega^2 LC V(z) \]
\[ \Rightarrow -k^2 V_+ e^{-j k z} = -\omega^2 LC V_+ e^{-j k z} \]
\[ \Rightarrow k^2 = \omega^2 LC \]
\[ \Rightarrow k = \omega \sqrt{LC} \]

Dispersion relation of a wave traveling with a velocity \( v = \frac{1}{\sqrt{LC}} \)

Impedance of a Transmission Line

Voltage is:
\[ V(z) = V_+ e^{-j k z} \]

Find the current from the transmission line equation:
\[ \frac{\partial V(z)}{\partial z} = -j \omega L I(z) \]
\[ \Rightarrow -j k V_+ e^{-j k z} = -j \omega L I(z) \]
\[ \Rightarrow I(z) = \frac{k}{\omega L} V_+ e^{-j k z} \]
\[ \Rightarrow I(z) = \frac{V_+}{Z_o} e^{-j k z} \]

Where \( Z_o \) given by:
\[ Z_o = \frac{\omega L}{k} = \sqrt{\frac{L}{C}} \]

is called the characteristic impedance of the transmission line

So a voltage-current wave propagating in the +z-direction on a transmission line is specified completely by:
\[ V(z) = V_+ e^{-j k z} \quad I(z) = \frac{V_+}{Z_o} e^{-j k z} \]
**Backward Waves on a Transmission Line**

A voltage-current wave propagating in the +z-direction on a transmission line is specified completely by:

\[ V(z) = V_+ e^{-jkz} \quad I(z) = I_+ e^{-jkz} = \frac{V_+}{Z_o} e^{-jkz} \]

A voltage-current wave propagating in the -z-direction on a transmission line is specified completely by:

\[ V(z) = V_- e^{+jkz} \quad I(z) = I_- e^{+jkz} = \frac{V_-}{Z_o} e^{+jkz} \]

Notice the –ve sign.

**Forward and Backward Waves on a Transmission Line**

In general, voltage on a transmission line is a superposition of forward and backward going waves:

\[ V(z) = V_+ e^{-jkz} + V_- e^{+jkz} \]

The corresponding current is also a superposition of forward and backward going waves:

\[ I(z) = I_+ e^{-jkz} + I_- e^{+jkz} = \frac{V_+}{Z_o} e^{-jkz} - \frac{V_-}{Z_o} e^{+jkz} \]
Parallel-Plate Transmission Lines: Fields, Voltages, and Currents

If the voltage and the current waves are:

\[ V(z) = V_+ e^{-j k z} \]
\[ I(z) = I_+ e^{-j k z} = \frac{V_+}{Z_0} e^{-j k z} \]

then the E-field and the H-field are (ignoring the fringing fields):

\[ E(z) = -\hat{y} \frac{V_+}{d} e^{-j k z} \]
\[ H(z) = \hat{x} \frac{I_+}{W} e^{-j k z} \]

Given the amplitude(s) of the voltage and/or current waves, the E-field and the H-field associated with the wave can be found.

Energy Flow and Power on a Transmission Line

Consider a transmission line with a voltage-current wave going in the +z-direction:

\[ \langle P_z(t) \rangle = \iint \langle \hat{S}(\hat{z}, t) \cdot \hat{z} \rangle \ dx dy \]
\[ = \iint \frac{1}{2} \text{Re} \left[ \hat{S}(\hat{r}) \cdot \hat{z} \right] \ dx dy \]

The area integral is over the entire x-y plane (or any plane parallel to the x-y plane).

It can be shown that this integral equals:

\[ \langle P_z(t) \rangle = \frac{1}{2} \text{Re} \left[ V_+ I_+^* \right] = \frac{1}{2} \text{Re} \left[ \frac{|V_+|^2}{Z_0} \right] \]

And if there is also a backward wave then:

\[ \langle P_z(t) \rangle = \frac{1}{2} \text{Re} \left[ V_+ I_+^* - V_- I_-^* \right] = \frac{1}{2} \text{Re} \left[ \frac{|V_+|^2}{Z_0} - \frac{|V_-|^2}{Z_0} \right] \]
Waves in Free Space and Waves in Transmission Lines

<table>
<thead>
<tr>
<th>Free Space</th>
<th>Transmission Lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nabla \times \mathbf{E}(\vec{r}) = -j \omega \mu_0 \mathbf{H}(\vec{r})$</td>
<td>$\frac{\partial V(z)}{\partial z} = -j \omega L I(z)$</td>
</tr>
<tr>
<td>$\nabla \times \mathbf{H}(\vec{r}) = j \omega \varepsilon_0 \mathbf{E}(\vec{r})$</td>
<td>$\frac{\partial I(z)}{\partial z} = -j \omega C V(z)$</td>
</tr>
<tr>
<td>$\nabla^2 \mathbf{E}(\vec{r}) = -\omega^2 \mu_0 \varepsilon_0 \mathbf{E}(\vec{r})$</td>
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</tr>
<tr>
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<td>$V(z) = V_0 e^{-j k z}$</td>
</tr>
<tr>
<td>$\mathbf{H}(\vec{r}) = \frac{y E_0}{\eta_0} e^{-j k z}$</td>
<td>$I(z) = \frac{V_0}{Z_0} e^{-j k z}$</td>
</tr>
</tbody>
</table>

Appendix: Energy Flow and Power on a Transmission Line

This appendix offers a proof of the power flow formula for arbitrary transmission lines

\[
\langle P_z(t) \rangle = \int \int \mathbf{S}(\vec{r}, t) \cdot \hat{z} \, dx \, dy \\
= \int \int \frac{1}{2} \text{Re} \left[ \mathbf{S}(\vec{r}) \right] \cdot \hat{z} \, dx \, dy \\
= \int \int \frac{1}{2} \text{Re} \left[ \mathbf{E}(\vec{r}) \times \mathbf{H}^*(\vec{r}) \right] \cdot \hat{z} \, dx \, dy
\]

By assumption both E- and H-fields have only transverse components (i.e. no component in the z-direction)

\[
\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} = \nabla_T + \hat{z} \frac{\partial}{\partial z}
\]

From Faraday's Law:

\[
\nabla \times \mathbf{E} = -j \omega \mu_0 \mathbf{H}
\]

\[
\Rightarrow \left( \nabla_T + \hat{z} \frac{\partial}{\partial z} \right) \times \mathbf{E} = -j \omega \mu_0 \mathbf{H}
\]
Therefore one may write the E-field as the transverse gradient of a scalar potential:

\[ \vec{E} = -\nabla_T \phi \]

Where by assumption the potential satisfies:

\[ \phi_{\text{1st conductor}} - \phi_{\text{2nd conductor}} = V_+ e^{-jkz} \]

From Ampere's Law:

\[ \nabla \times \vec{H} = \hat{z} J_z + j \omega \epsilon \vec{E} \]

\[ \Rightarrow \left( \nabla_T + \hat{z} \frac{\partial}{\partial z} \right) \times \vec{H} = \hat{z} J_z + j \omega \epsilon \vec{E} \]

\[ \Rightarrow \nabla_T \times \vec{H} = \hat{z} J_z \]

\[ \int_{\text{1st conductor}} J_z \ dx \ dy \] = \[ I_+ e^{-jkz} \]

\[ \int_{\text{2nd conductor}} J_z \ dx \ dy \] = \[ -I_+ e^{-jkz} \]

\[ P_z(t) = \int \frac{1}{2} \text{Re}[\vec{E} \times \vec{H}^*] \cdot \hat{z} \ dx \ dy \]

\[ = \frac{1}{2} \text{Re} \left[ \left( -\nabla_T \phi \times \vec{H} \right) \cdot \hat{z} \right] \ dx \ dy \]

\[ = \frac{1}{2} \text{Re} \left[ \left( \phi_{\text{1st conductor}} - \phi_{\text{2nd conductor}} \right) \hat{z} \right] \ dx \ dy \]

\[ = \frac{1}{2} \text{Re}[\left[ \phi_{\text{1st conductor}} - \phi_{\text{2nd conductor}} \right] I_+ e^{jkz}] \]

\[ = \frac{1}{2} \text{Re}[V_+ I_+] \]