Lecture 20

Transmission Lines: The Basics

In this lecture you will learn:

- Transmission lines
- Different types of transmission line structures
- Transmission line equations
- Power flow in transmission lines
- Appendix

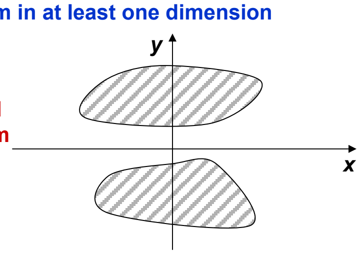
Guided Waves

- So far in the course you have been dealing with waves that propagated in infinite size media
- For many applications it is desirable to have electromagnetic energy be guided in much the same way as water flow is guided by having it flow in pipes
- Transmission lines are the simplest structures that guide electromagnetic waves

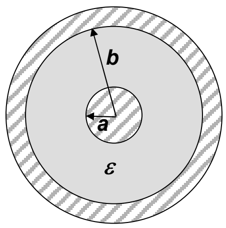
Transmission line:

In this course a transmission line would be any two arbitrary shaped metal conductors that are very long and uniform in at least one dimension

A transmission line made up of two metal conductors that are very long and uniform in the *z*-direction



Common Transmission Line Structures - I

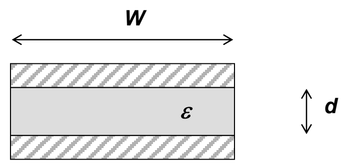


Co-axial Cable

$$C = \frac{2\pi\,\varepsilon}{\log\left(\frac{b}{a}\right)}$$

$$L = \frac{\mu_o}{2\pi}\log\left(\frac{b}{a}\right)$$

$$LC = \mu_o\,\varepsilon$$



Parallel-Plate Transmission Line

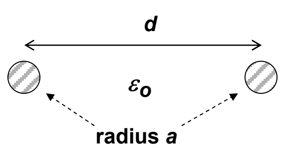
$$C = \varepsilon\frac{W}{d}$$

$$L = \mu_o\frac{d}{W}$$

$$LC = \mu_o\,\varepsilon$$

Capacitance and Inductance per unit length for each structure are shown

Common Transmission Line Structures - II



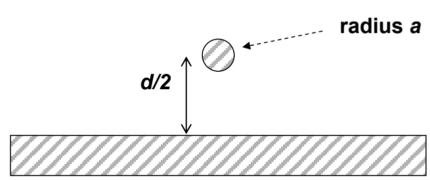
Parallel Metal Wires

For $d >> a$:

$$C = \frac{\pi\,\varepsilon_o}{\log\left(\frac{d}{a}\right)}$$

$$L = \frac{\mu_o}{\pi}\log\left(\frac{d}{a}\right)$$

$$LC = \mu_o\,\varepsilon_o$$



Metal Wire Over a Ground Plane

For $d >> a$:

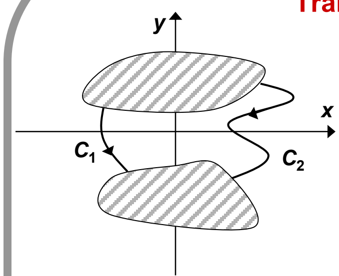
$$C = \frac{2\pi\,\varepsilon_o}{\log\left(\frac{d}{a}\right)}$$

$$L = \frac{\mu_o}{2\pi}\log\left(\frac{d}{a}\right)$$

$$LC = \mu_o\,\varepsilon_o$$

Capacitance and Inductance per unit length for each structure are shown

Transmission Line Voltages



- Suppose the potential difference between the two conductors of a transmission line at location *z* is *V*(*z*) then E-field line integral in a plane parallel to the *x-y* plane at the location *z* is independent of the contour taken, and is related to the potential difference *V*(*z*) as:

$$V(z) = \int_{c_1} \vec{E}.d\vec{s} = \int_{c_2} \vec{E}.d\vec{s}$$

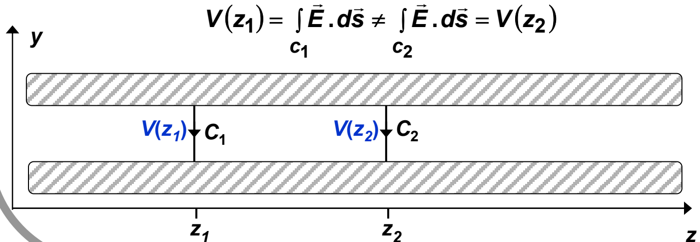
- The charge per unit length *Q*(*z*) on the transmission line at location *z* is:

$$Q(z) = C\,V(z)$$

Capacitance per unit length

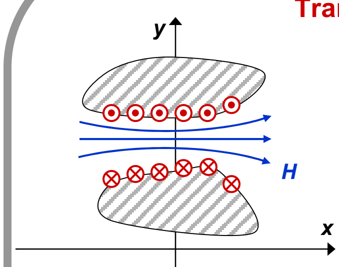
Two points with different *z*-values can have different potentials:

$$V(z_1) = \int_{c_1} \vec{E}.d\vec{s} \neq \int_{c_2} \vec{E}.d\vec{s} = V(z_2)$$



The conductors are no longer equi-potentials

Transmission Line Currents

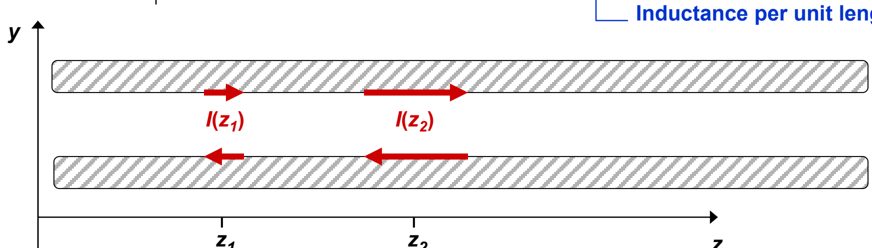


- Suppose the total current in the upper conductor in the +*z*-direction at location *z* is *I*(*z*) and in the lower conductor is –*I*(*z*)

- The H-field flux per unit length $\lambda(z)$ enclosed between the two conductors at the location *z* is related to the current *I*(*z*) as:

$$\lambda(z) = L\,I(z)$$

Inductance per unit length



Two points with different *z*-values can have different currents

$$I(z_1) \neq I(z_2)$$

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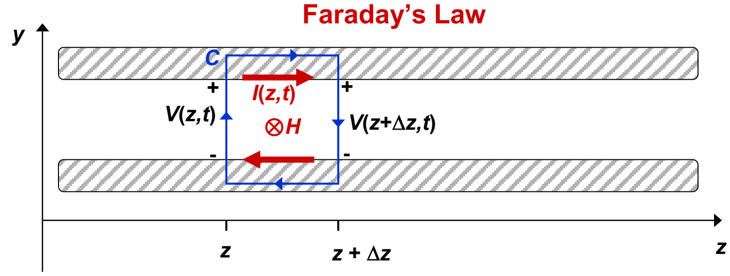
Faraday's Law



Use Faraday's law for the contour C:

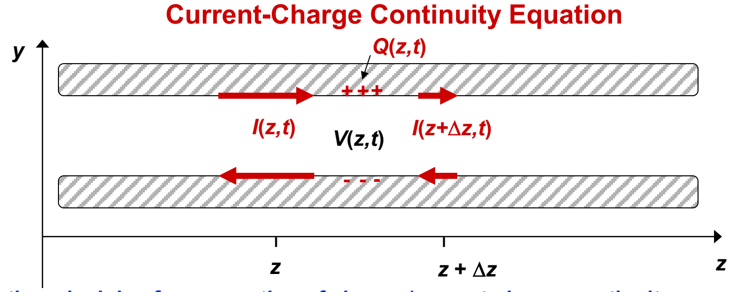
$$\oint_c \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t}\iint \mu_o \vec{H} \cdot d\vec{a}$$

$$\Rightarrow \quad V(z+\Delta z,t) - V(z,t) = -\frac{\partial \lambda(z,t)\,\Delta z}{\partial t} \longrightarrow \left\{ \lambda(z,t) = L\,I(z,t) \right.$$

$$\Rightarrow \quad \frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -\frac{\partial L\,I(z,t)}{\partial t}$$

$$\Rightarrow \quad \boxed{\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t}} \longrightarrow (1)$$

Current-Charge Continuity Equation



Use the principle of conservation of charge (current-charge continuity equation):

If current is varying in space, there must be charge either piling up or piling down somewhere

$$I(z,t) - I(z+\Delta z,t) = \frac{\partial Q(z,t)\,\Delta z}{\partial t} \longrightarrow \left\{ Q(z,t) = C\,V(z,t) \right.$$

$$\Rightarrow \quad \frac{I(z+\Delta z,t) - I(z,t)}{\Delta z} = -\frac{\partial C\,V(z,t)}{\partial t}$$

$$\Rightarrow \quad \boxed{\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t}} \longrightarrow (2)$$

Transmission Line Equations

The following two equation describe the propagation of guided electromagnetic waves on transmission lines (also called the Telegrapher's Equations):

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t} \qquad (1)$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t} \qquad (2)$$

Wave equations:

$$(1) \Rightarrow \frac{\partial^2 V(z,t)}{\partial z^2} = -L\frac{\partial^2 I(z,t)}{\partial z\,\partial t} \qquad (3)$$

$$(2) \Rightarrow \frac{\partial^2 I(z,t)}{\partial t\,\partial z} = -C\frac{\partial^2 V(z,t)}{\partial t^2} \qquad (4)$$

$$(3) \text{ and } (4) \Rightarrow \boxed{\frac{\partial^2 V(z,t)}{\partial z^2} = LC\frac{\partial^2 V(z,t)}{\partial t^2}}$$

→ Equation of a wave traveling with a velocity $= v = 1/\sqrt{LC}$

A similar equation can be derived for the current:

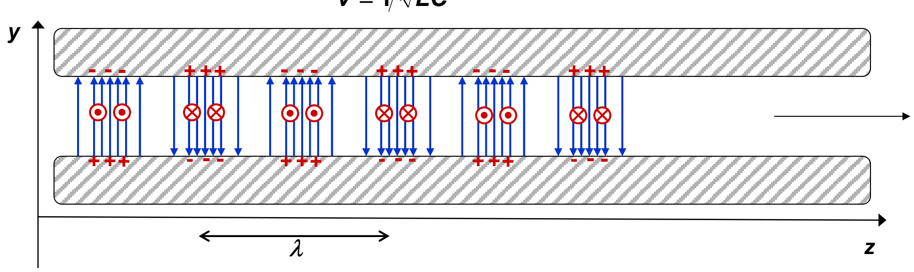
$$\boxed{\frac{\partial^2 I(z,t)}{\partial z^2} = LC\frac{\partial^2 I(z,t)}{\partial t^2}}$$

Nature of Guided Waves in Transmission Lines - I

$$\frac{\partial^2 V(z,t)}{\partial z^2} = LC\frac{\partial^2 V(z,t)}{\partial t^2} \qquad \frac{\partial^2 I(z,t)}{\partial z^2} = LC\frac{\partial^2 I(z,t)}{\partial t^2}$$

The guided wave consists of E-fields and H-fields together with charges and currents on the conductors that all move together in sync with a velocity given by:

$$v = 1/\sqrt{LC}$$



• The charges satisfy the boundary conditions for the E-fields

Nature of Guided Waves in Transmission Lines - II

$$\frac{\partial^2 V(z,t)}{\partial z^2} = LC\frac{\partial^2 V(z,t)}{\partial t^2} \qquad \frac{\partial^2 I(z,t)}{\partial z^2} = LC\frac{\partial^2 I(z,t)}{\partial t^2}$$

The guided wave consists of E-fields and H-fields together with charges and currents on the conductors that all move together in sync with a velocity given by:

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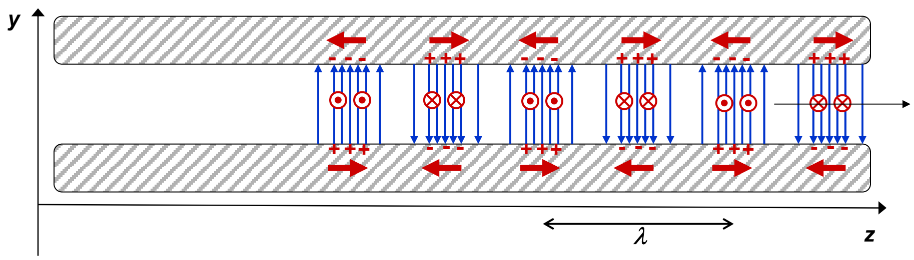
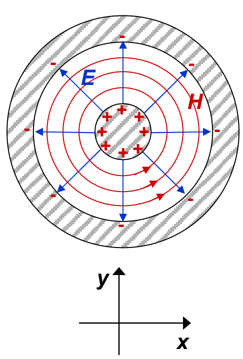
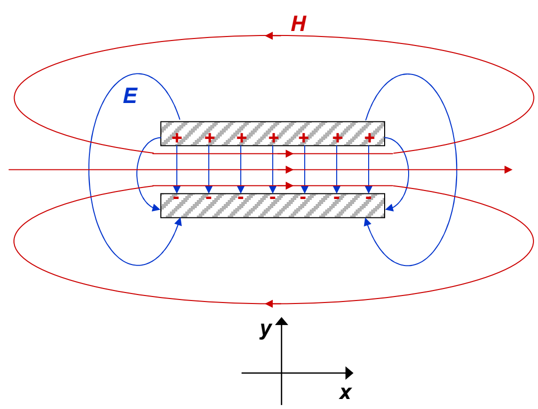


- **The charges satisfy the boundary conditions for the E-fields**
- **The currents satisfy the boundary conditions for the H-fields**
- **The wave is called a TEM wave since both the E-field and H-field point in a direction transverse to the direction of propagation**

E-Fields and H-fields for Common Transmission Lines



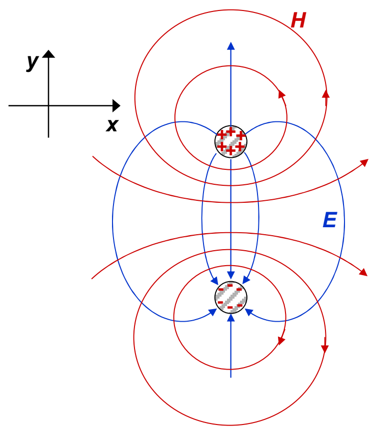
Co-axial Cable



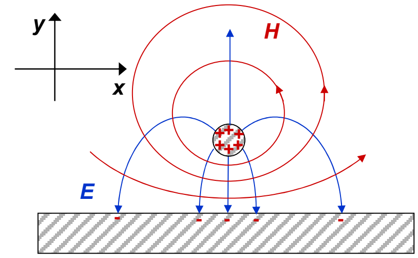
Parallel-Plate Transmission Line

Notice that at each point $\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$ points in the +z-direction – indicating energy flow in the +z-direction

E-Fields and H-fields for Common Transmission Lines



Parallel Metal Wires



Metal Wire Over a Ground Plane

Notice that at each point $\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$ points in the +*z*-direction – indicating energy flow in the +*z*-direction

Current and Voltage Phasors

Convert to phasors:

$$V(z,t) = \mathrm{Re}\left[V(z)\, e^{j\omega t}\right]$$

$$I(z,t) = \mathrm{Re}\left[I(z)\, e^{j\omega t}\right]$$

Transmission line equations in phasor notation:

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t} \longrightarrow \frac{\partial V(z)}{\partial z} = -j\omega L\, I(z)$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t} \longrightarrow \frac{\partial I(z)}{\partial z} = -j\omega C\, V(z)$$

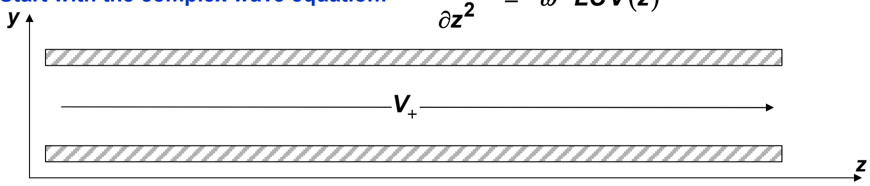
Wave equations in phasor notation:

$$\frac{\partial^2 V(z,t)}{\partial z^2} = LC\frac{\partial^2 V(z,t)}{\partial t^2} \longrightarrow \frac{\partial^2 V(z)}{\partial z^2} = -\omega^2 LC\, V(z)$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} = LC\frac{\partial^2 I(z,t)}{\partial t^2} \longrightarrow \frac{\partial^2 I(z)}{\partial z^2} = -\omega^2 LC\, I(z)$$

Solutions of Transmission Line Equations

Start with the complex wave equation: $\dfrac{\partial^2 V(z)}{\partial z^2} = -\omega^2 LC\, V(z)$



Assume a solution of the form of a traveling wave: $V(z) = V_+\, e^{-jkz}$

A wave traveling in the +*z*-direction

Substitute in the complex wave equation:

$$\frac{\partial^2 V(z)}{\partial z^2} = -\omega^2 LC\, V(z)$$

$$\Rightarrow \quad -k^2 V_+\, e^{-jkz} = -\omega^2\, LC\, V_+\, e^{-jkz}$$

$$\Rightarrow \quad k^2 = \omega^2\, LC$$

$$\Rightarrow \quad k = \omega\sqrt{LC}$$

Dispersion relation of a wave traveling with a velocity $= v = 1/\sqrt{LC}$

Impedance of a Transmission Line

Voltage is: $V(z) = V_+\, e^{-jkz}$

Find the current from the transmission line equation:

$$\frac{\partial V(z)}{\partial z} = -j\,\omega L\, I(z)$$

$$\Rightarrow \quad -jk V_+\, e^{-jkz} = -j\,\omega\, L\, I(z)$$

$$\Rightarrow \quad I(z) = \frac{k}{\omega L} V_+\, e^{-jkz}$$

$$\Rightarrow \quad I(z) = \frac{V_+}{Z_o}\, e^{-jkz}$$

Where Z_o, given by: $Z_o = \dfrac{\omega L}{k} = \sqrt{\dfrac{L}{C}}$

is called the characteristic impedance of the transmission line

So a voltage-current wave propagating in the +*z*-direction on a transmission line is specified completely by:

$$V(z) = V_+\, e^{-jkz} \qquad\qquad I(z) = I_+\, e^{-jkz} = \frac{V_+}{Z_o}\, e^{-jkz}$$

Backward Waves on a Transmission Line

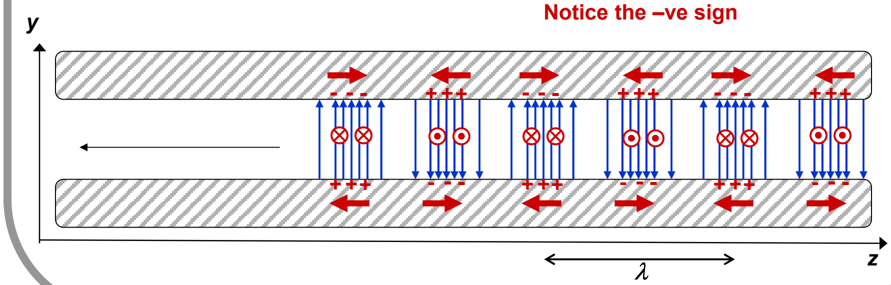
A voltage-current wave propagating in the +*z*-direction on a transmission line is specified completely by:

$$V(z) = V_+ \, e^{-jkz} \qquad I(z) = I_+ \, e^{-jkz} = \frac{V_+}{Z_o} e^{-jkz}$$

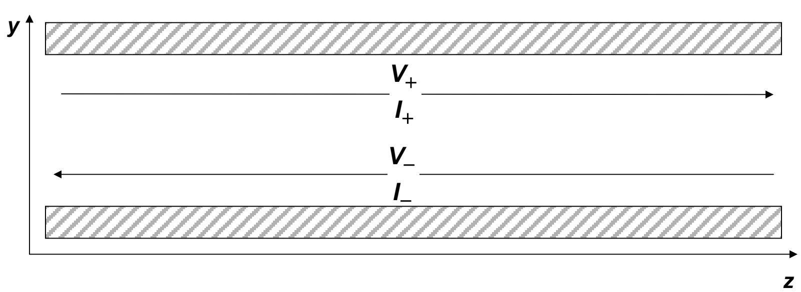
A voltage-current wave propagating in the -*z*-direction on a transmission line is specified completely by:

$$V(z) = V_- \, e^{+jkz} \qquad I(z) = I_- \, e^{+jkz} = -\frac{V_-}{Z_o} e^{+jkz}$$

Notice the –ve sign



Forward and Backward Waves on a Transmission Line



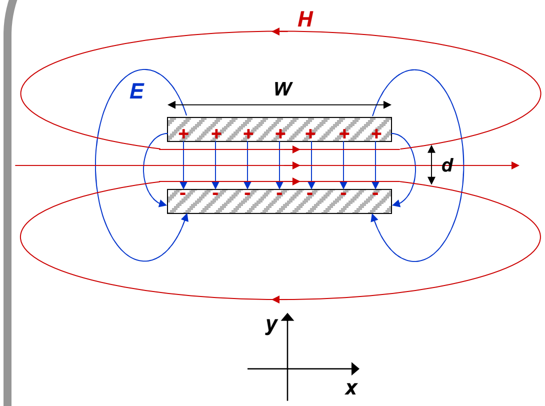
In general, voltage on a transmission line is a superposition of forward and backward going waves:

$$V(z) = V_+ \, e^{-jkz} + V_- \, e^{+jkz}$$

The corresponding current is also a superposition of forward and backward going waves:

$$I(z) = I_+ \, e^{-jkz} + I_- \, e^{+jkz} = \frac{V_+}{Z_o} e^{-jkz} - \frac{V_-}{Z_o} e^{+jkz}$$

Parallel-Plate Transmission Lines: Fields, Voltages, and Currents



Parallel-Plate Transmission Line

If the voltage and the current waves are:

$$V(z) = V_+ \, e^{-jkz}$$

$$I(z) = I_+ \, e^{-jkz} = \frac{V_+}{Z_o} \, e^{-jkz}$$

then the E-field and the H-field are (ignoring the fringing fields):

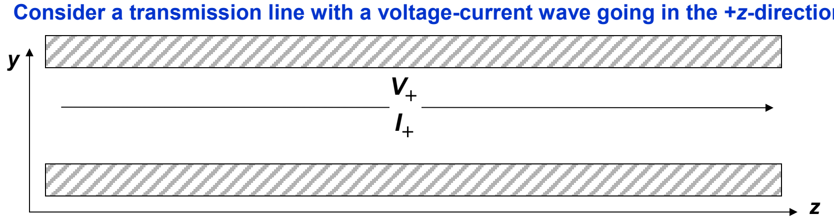
$$\Rightarrow \quad \vec{E}(z) = -\hat{y} \, \frac{V_+}{d} \, e^{-jkz}$$

$$\Rightarrow \quad \vec{H}(z) = \hat{x} \, \frac{I_+}{W} \, e^{-jkz}$$

Given the amplitude(s) of the voltage and/or current waves, the E-field and the H-field associated with the wave can be found

Energy Flow and Power on a Transmission Line

Consider a transmission line with a voltage-current wave going in the +z-direction:



How much is the total time-average power (not power per unit area) carried by the wave in the +z-direction?

$$\langle P_z(t) \rangle = \iint \langle \vec{S}(\vec{r},t) \rangle . \hat{z} \quad dxdy$$

The area integral is over the entire *x*-*y* plane (or any plane parallel to the *x*-*y* plane)

$$= \iint \frac{1}{2} \mathrm{Re}\left[\vec{S}(\vec{r}) \right] . \hat{z} \quad dxdy$$

It can be show that this integral equals: $\langle P_z(t) \rangle = \frac{1}{2} \mathrm{Re}\left[V_+ \, I_+^* \right] = \frac{1}{2} \mathrm{Re}\left[\frac{|V_+|^2}{Z_o^*} \right]$

And if there is a also backward wave then:

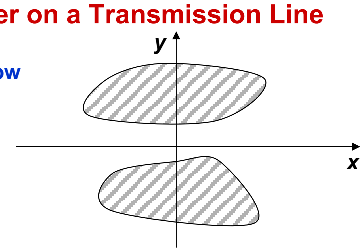
$$\langle P_z(t) \rangle = \frac{1}{2} \mathrm{Re}\left[V_+ \, I_+^* - V_- \, I_-^* \right] = \frac{1}{2} \mathrm{Re}\left[\frac{|V_+|^2}{Z_o^*} - \frac{|V_-|^2}{Z_o^*} \right]$$

Waves in Free Space and Waves in Transmission Lines

Free Space	Transmission Lines
$\nabla \times \vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\vec{H}(\vec{r})$	$\dfrac{\partial V(z)}{\partial z} = -j\,\omega L\,I(z)$
$\nabla \times \vec{H}(\vec{r}) = j\,\omega\,\varepsilon_o\,\vec{E}(\vec{r})$	$\dfrac{\partial I(z)}{\partial z} = -j\,\omega C\,V(z)$
$\nabla^2 \vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon_o\,\vec{E}(\vec{r})$	$\dfrac{\partial^2 V(z)}{\partial z^2} = -\omega^2\,LC\,V(z)$
$\nabla^2 \vec{H}(\vec{r}) = -\omega^2\,\mu_o\,\varepsilon_o\,\vec{H}(\vec{r})$	$\dfrac{\partial^2 I(z)}{\partial z^2} = -\omega^2\,LC\,I(z)$
$\vec{E}(\vec{r}) = \hat{x}\,E_o\,e^{-j\,k\,z}$	$V(z) = V_+\,e^{-j\,k\,z}$
$\vec{H}(\vec{r}) = \hat{y}\,\dfrac{E_o}{\eta_o}\,e^{-j\,k\,z}$	$I(z) = \dfrac{V_+}{Z_o}\,e^{-j\,k\,z}$

Appendix: Energy Flow and Power on a Transmission Line

This appendix offers a proof of the power flow formula for arbitrary transmission lines



$$\langle P_z(t)\rangle = \iint \langle \vec{S}(\vec{r},t)\rangle.\hat{z}\quad dxdy$$
$$= \iint \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right].\hat{z}\quad dxdy$$
$$= \iint \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right].\hat{z}\quad dxdy$$

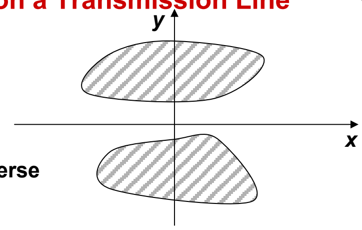
By assumption both E- and H-fields have only transverse components (i.e. no component in the z-direction)

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \nabla_T + \hat{z}\frac{\partial}{\partial z}$$

From Faraday's Law:

$$\nabla \times \vec{E} = -j\omega\,\mu_o\,\vec{H}$$
$$\Rightarrow \left(\nabla_T + \hat{z}\frac{\partial}{\partial z}\right)\times\vec{E} = -j\omega\,\mu_o\vec{H}$$

Appendix: Energy Flow and Power on a Transmission Line



$$\Rightarrow \quad \left(\nabla_T + \hat{z}\frac{\partial}{\partial z}\right)\times\vec{E} = -j\omega\,\mu_o\vec{H}$$

$$\Rightarrow \quad \nabla_T \times \vec{E} = 0$$

Therefore one may write the E-field as the transverse gradient of a scalar potential:

$$\vec{E} = -\nabla_T\phi$$

Where by assumption the potential satisfies:

$$\phi\big|_{1^{st}\text{ conductor}} - \phi\big|_{2^{nd}\text{ conductor}} = V_+\, e^{-jkz}$$

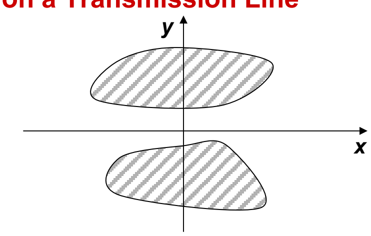
From Amperes's Law:

$$\nabla\times\vec{H} = \hat{z}\,J_z + j\omega\,\varepsilon\,\vec{E}$$

$$\Rightarrow \quad \left(\nabla_T + \hat{z}\frac{\partial}{\partial z}\right)\times\vec{H} = \hat{z}\,J_z + j\omega\,\varepsilon\,E$$

$$\Rightarrow \quad \nabla_T\times\vec{H} = \hat{z}\,J_z$$

Appendix: Energy Flow and Power on a Transmission Line



$$\iint_{1^{st}\text{ conductor}} J_z \;\; dxdy = I_+\, e^{-jkz}$$

$$\iint_{2^{nd}\text{ conductor}} J_z \;\; dxdy = -I_+\, e^{-jkz}$$

$$\langle P_z(t)\rangle = \iint \frac{1}{2}\text{Re}\left[\vec{E}\times\vec{H}^*\right].\hat{z} \;\; dxdy$$

$$= \frac{1}{2}\text{Re}\iint\left[-\nabla_T\phi\times\vec{H}^*\right].\hat{z}\;\; dxdy$$

$$= \frac{1}{2}\text{Re}\iint \underbrace{\nabla_T\times\left[-\phi\,\vec{H}^*\right].\hat{z}}_{0}\;\; dxdy$$

$$+\frac{1}{2}\text{Re}\iint\left[\phi\,\nabla_T\times\vec{H}^*\right].\hat{z}\;\; dxdy$$

$$= \frac{1}{2}\text{Re}\iint\left[\phi\,J_z^*\right]\;dxdy$$

$$= \frac{1}{2}\text{Re}\left[\left(\phi\big|_{1^{st}\text{ conductor}} - \phi\big|_{2^{nd}\text{ conductor}}\right)I_+^*\, e^{jkz}\right]$$

$$= \frac{1}{2}\text{Re}\left[V_+\, I_+^*\right]$$