Lecture 2

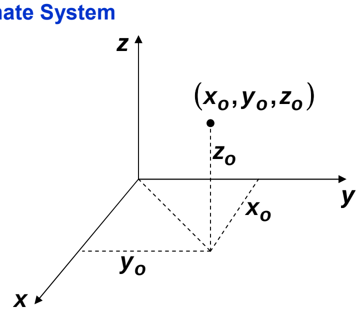
Maxwell's Equations in Free Space

In this lecture you will learn:

- Co-ordinate Systems and Course Notations
- Maxwell's Equations in Differential and Integral Forms
- Electrostatics and Magnetostatics
- Electroquasistatics and Magnetoquasistatics

Co-ordinate Systems and Vectors

Cartesian Coordinate System



Vectors in Cartesian Coordinate System

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

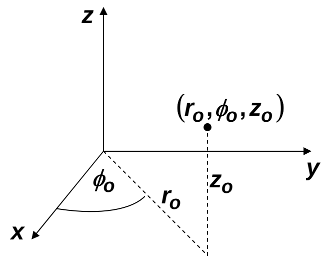
$$\vec{A} = A_x \hat{i}_x + A_y \hat{i}_y + A_z \hat{i}_z$$

→ All mean exactly the same thing … just a different notation for the unit vectors

The first one will be used in this course

Co-ordinate Systems and Vectors

Cylindrical Coordinate System



Vectors in Cylindrical Coordinate System

$$\vec{A} = A_r\,\hat{r} + A_\phi\,\hat{\phi} + A_z\,\hat{z}$$

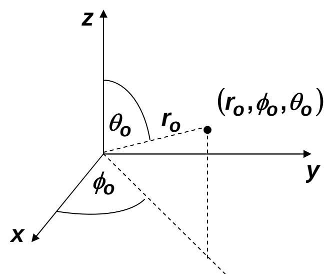
$$\vec{A} = A_r\,\hat{i}_r + A_\phi\,\hat{i}_\phi + A_z\,\hat{i}_z$$

→ Both mean exactly the same thing … just a different notation for the unit vectors

The first one will be used in this course

Co-ordinate Systems and Vectors

Spherical Coordinate System



Vectors in Spherical Coordinate System

$$\vec{A} = A_r\,\hat{r} + A_\phi\,\hat{\phi} + A_\theta\,\hat{\theta}$$

$$\vec{A} = A_r\,\hat{i}_r + A_\phi\,\hat{i}_\phi + A_\theta\,\hat{i}_\theta$$

→ Both mean exactly the same thing … just a different notation for the unit vectors

The first one will be used in this course

Vector Fields

In layman terms, a vector field implies a vector associated with every point is space:

Examples:

Electric Field: $\vec{E}(x,y,z,t)$ or $\vec{E}(\vec{r},t)$

Magnetic Field: $\vec{H}(x,y,z,t)$ or $\vec{H}(\vec{r},t)$

Maxwell's Equations in Free Space

Integral Form		Differential Form
(1) $\oiint \varepsilon_o \vec{E}\,.\,d\vec{a} = \iiint \rho\, dV$	← Gauss' Law →	$\nabla.\,\varepsilon_o \vec{E} = \rho$
(2) $\oiint \mu_o \vec{H}\,.\,d\vec{a} = 0$	← Gauss' Law →	$\nabla.\,\mu_o \vec{H} = 0$
(3) $\oint \vec{E}\,.\,d\vec{s} = -\frac{\partial}{\partial t}\iint \mu_o \vec{H}.\,d\vec{a}$	← Faraday's Law →	$\nabla \times \vec{E} = -\frac{\partial\, \mu_o \vec{H}}{\partial t}$
(4) $\oint \vec{H}\,.\,d\vec{s} = \iint \vec{J}.d\vec{a} + \frac{\partial}{\partial t}\iint \varepsilon_o \vec{E}\,.\,d\vec{a}$	← Ampere's Law →	$\nabla \times \vec{H} = \vec{J} + \frac{\partial\, \varepsilon_o \vec{E}}{\partial t}$

Lorentz Force Law

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \mu_o \vec{H}\right)$$

Lorentz Law describes the effect of electromagnetic fields upon charges

Physical Quantities, Values, and SI Units

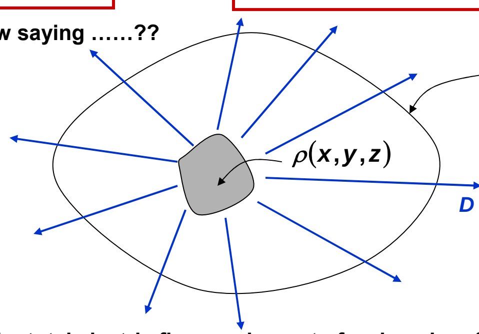
	Quantity	Value/Units
$\vec{E}$	Electric Field	Volts/m
$\vec{H}$	Magnetic Field	Amps/m
ε_o	Permittivity of Free Space	$8.85\text{x}10^{-12}$ Farads/m
μ_o	Permeability of Free Space	$4\pi\text{x}10^{-7}$ Henry/m
q	Electronic Unit of Charge	$1.6\text{x}10^{-19}$ Coulombs
ρ	Volume Charge Density	Coulombs/m^3
$\vec{J}$	Current Density	Amps/m^2
$\vec{D} = \varepsilon_o \vec{E}$	Electric Flux Density	Coulombs/m^2
$\vec{B} = \mu_o \vec{H}$	Magnetic Flux Density	Tesla

Gauss' Law – Integral Form

$$\oiint \varepsilon_o \vec{E} \cdot d\vec{a} = \iiint \rho \, dV$$ or $$\oiint \vec{D} \cdot d\vec{a} = \iiint \rho \, dV$$

What is this law saying ……??

Carl F. Gauss
(1777-1855)



A closed surface of arbitrary shape surrounding a charge distribution

Gauss' Law: The total electric flux coming out of a closed surface is equal to the total charge enclosed by that closed surface (irrespective of the shape of the closed surface)

Points to Note: This law establishes charges as the "sources" or "sinks" of the electric field (i.e. charges produce or terminate electric field lines).

If the total flux through a closed surface is positive, then the total charge enclosed is positive. If the total flux is negative, then the total charge enclosed is negative

Gauss' Law – Differential Form

Divergence Theorem:

For any vector field: $\oiint \vec{A} \cdot d\vec{a} = \iiint \nabla \cdot \vec{A}\; dV$

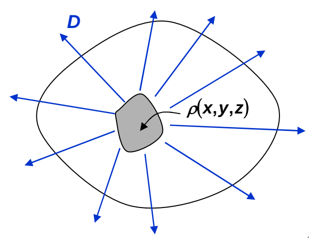
The flux of a vector through a closed surface is equal to the integral of the divergence of the vector taken over the volume enclosed by that closed surface

Using the Divergence Theorem with Gauss' Law in Integral Form:

$$\oiint \vec{D} \cdot d\vec{a} = \iiint \rho \, dV$$

$$\Rightarrow \iiint \nabla \cdot \vec{D}\, dV = \iiint \rho \, dV$$

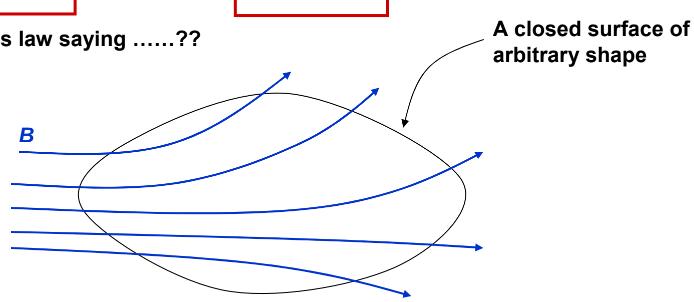
$$\Rightarrow \nabla \cdot \vec{D} = \rho \qquad \text{or} \qquad \nabla \cdot \varepsilon_o \vec{E} = \rho$$



Gauss' Law for the Magnetic Field – Integral Form

$$\oiint \mu_o \vec{H} \cdot d\vec{a} = 0 \qquad \text{or} \qquad \oiint \vec{B} \cdot d\vec{a} = 0$$

What is this law saying ……??



Gauss' Law for the Magnetic Fields: The total magnetic flux coming out of a closed surface is always zero.

Points to Note: This law implies that there are no such things as "magnetic charges" that can emanate or terminate magnetic field lines.

If magnetic field is non-zero, then the flux into any closed surface must equal the flux out of it - so that the net flux coming out is zero.

Gauss' Law – Differential Form

Divergence Theorem:

For any vector field: $\oiint \vec{A}\,.\,d\vec{a} = \iiint \nabla\,.\,\vec{A}\;dV$

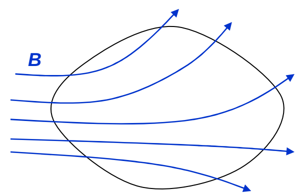
The flux of a vector through a closed surface is equal to the integral of the divergence of the vector taken over the volume enclosed by that closed surface

Using the Divergence Theorem with Gauss' Law for the Magnetic Field in Integral form:

$\oiint \vec{B}\,.\,d\vec{a} = 0$ **(Remember** $\vec{B} = \mu_o \vec{H}$ **)**

$\Rightarrow \iiint \nabla\,.\vec{B}\,dV = 0$

$\Rightarrow \nabla\,.\vec{B} = 0$ **or** $\nabla\,.\,\mu_o \vec{H} = 0$

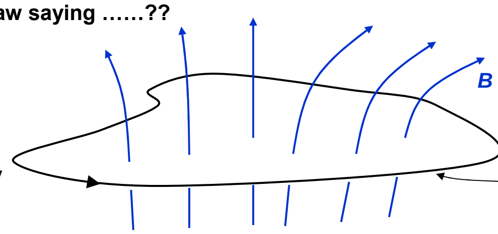


Faraday's Law – Integral Form

$\oint \vec{E}\,.\,d\vec{s} = -\dfrac{\partial}{\partial t}\iint \mu_o\vec{H}\,.\,d\vec{a}$ **or** $\oint \vec{E}\,.\,d\vec{s} = -\dfrac{\partial}{\partial t}\iint \vec{B}\,.\,d\vec{a}$

What is this law saying ……??

Michael Faraday (1791-1867)



Faraday's Law: The line integral of electric field over a closed contour is equal to –ve of the time rate of change of the total magnetic flux that goes through any arbitrary surface that is bounded by the closed contour

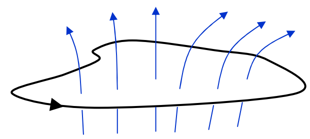
Points to Note: This law says that time changing magnetic fields can also generate electric fields

The positive directions for the surface normal vector and of the contour are related by the right hand rule

Faraday's Law – Differential Form

Stokes Theorem:

For any vector field: $\oint \vec{A}.d\vec{s} = \iint \left(\nabla \times \vec{A}\right).\,d\vec{a}$



The line integral of a vector over a closed contour is equal to the surface integral of the curl of that vector over any arbitrary surface that is bounded by the closed contour

Using the Stokes Theorem with Farady's Law in Integral Form:

$$\oint \vec{E}\,.\,d\vec{s} = -\frac{\partial}{\partial t}\iint \vec{B}\,.\,d\vec{a}$$

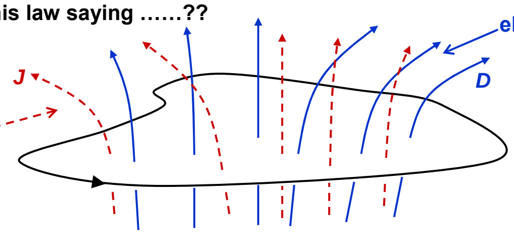
$$\Rightarrow \iint \left(\nabla \times \vec{E}\right).d\vec{a} = -\frac{\partial}{\partial t}\iint \vec{B}\,.\,d\vec{a}$$

$$\Rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{or} \quad \nabla \times \vec{E} = -\frac{\partial\, \mu_o \vec{H}}{\partial t}$$

Ampere's Law – Integral Form

$$\oint \vec{H}\,.\,d\vec{s} = \iint \vec{J}.d\vec{a} + \frac{\partial}{\partial t}\iint \varepsilon_o \vec{E}\,.\,d\vec{a} \quad \text{or} \quad \oint \vec{H}\,.\,d\vec{s} = \iint \vec{J}.d\vec{a} + \frac{\partial}{\partial t}\iint \vec{D}.d\vec{a}$$

What is this law saying ……??



A. M. Ampere
(1775-1836)

Ampere's Law: The line integral of magnetic field over a closed contour is equal to the total current plus the time rate of change of the total electric flux that goes through any arbitrary surface that is bounded by the closed contour

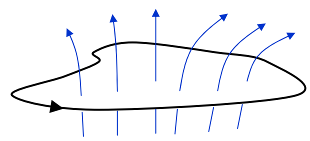
Points to Note: This law says that electrical currents and time changing electric fields can generate magnetic fields. Since there are no magnetic charges, this is the only known way to generate magnetic fields

The positive directions for the surface normal vector and of the contour are related by the right hand rule

Ampere's Law – Differential Form

Stokes Theorem:

For any vector field: $\oint \vec{A}.d\vec{s} = \iint \left(\nabla \times \vec{A}\right).d\vec{a}$



The line integral of a vector over a closed contour is equal to the surface integral of the curl of that vector over any arbitrary surface that is bounded by the closed contour

Using the Stokes Theorem with Ampere's Law in Integral Form:

$$\oint \vec{H}.d\vec{s} = \iint \vec{J}.d\vec{a} + \frac{\partial}{\partial t}\iint \vec{D}.d\vec{a}$$

$$\Rightarrow \iint \left(\nabla \times \vec{H}\right).d\vec{a} = \iint \vec{J}.d\vec{a} + \frac{\partial}{\partial t}\iint \vec{D}.d\vec{a}$$

$$\Rightarrow \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{or} \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \varepsilon_o \vec{E}}{\partial t}$$

Maxwell's Equations and Light – Coupling of *E* and *H* Fields

$$\nabla.\varepsilon_o \vec{E} = \rho \qquad \nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t}$$

$$\nabla.\mu_o \vec{H} = 0 \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \varepsilon_o \vec{E}}{\partial t}$$

Time varying electric and magnetic fields are coupled - this coupling is responsible for the propagation of electromagnetic waves

Electromagnetic Wave Equation in Free Space:

Assume: $\rho = \vec{J} = 0$ and take the curl of the Faraday's Law on both sides:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla \times \left(\frac{\partial \mu_o \vec{H}}{\partial t}\right) = -\frac{\partial \mu_o \nabla \times \vec{H}}{\partial t} = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \nabla \times \left(\nabla \times \vec{E}\right) = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \approx 3 \times 10^8 \text{ m/s}$$

$$\Rightarrow \nabla \times \left(\nabla \times \vec{E}\right) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow$$

Equation for a wave traveling at the speed *c*

ECE 303 – Fall 2007 – Farhan Rana – Cornell University

Maxwell's Equations and Light

$$\nabla \times (\nabla \times \vec{E}) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \longrightarrow$$

Equation for a wave traveling at the speed c:

$$c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \approx 3 \times 10^8 \text{ m/s}$$

In 1865 Maxwell wrote:

> *"This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws"*

Electrostatics and Magnetostatics

Suppose we restrict ourselves to time-independent situations (i.e. nothing is varying with time – everything is stationary)

We get two sets of equations for electric and magnetic fields that are completely independent and uncoupled:

Equations of Electrostatics

$$\nabla . \varepsilon_o \vec{E}(\vec{r}) = \rho(\vec{r})$$

$$\nabla \times \vec{E}(\vec{r}) = 0$$

• Electric fields are produced by only electric charges

• In electrostatics problems one needs to determine electric field given some charge distribution

Equations of Magnetostatics

$$\nabla . \mu_o \vec{H}(\vec{r}) = 0$$

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}$$

• Magnetic fields are produced by only electric currents

• In magnetostatics problems one needs to determine magnetic field given some current distribution

Electroquasistatics and Magnetoquasistatics - I

• The restriction to completely time-independent situations is too limiting and often unnecessary

• What if things are changing in time but "slowly" ……..(how slowly is "slowly" ?)

• Allowing for slow time variations, one often uses the equations of electroquasistatics and magnetoquasistatics

Equations of Electroquasistatics

$$\nabla \cdot \varepsilon_o \vec{E}(\vec{r},t) = \rho(\vec{r},t)$$

$$\nabla \times \vec{E}(\vec{r},t) = 0$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \varepsilon_o \vec{E}}{\partial t}$$

• Electric fields are produced by only electric charges

• Once the electric field is determined, the magnetic field can be found by the last equation

Equations of Magnetoquasistatics

$$\nabla \cdot \mu_o \vec{H}(\vec{r},t) = 0$$

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t)$$

$$\nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t}$$

• Magnetic fields are produced by only electric currents

• Once the magnetic field is determined, the electric field can be found by the last equation

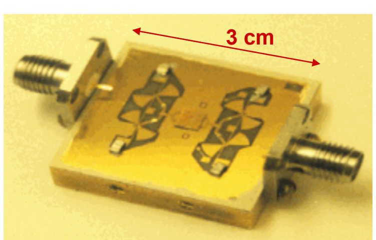
Electroquasistatics and Magnetoquasistatics - II

Question from the last slide: How slowly is "slowly" ?

• Electromagnetic waves and signals move at the speed *c* (speed of light)

Answer: Time variations are considered slow if the time scales over which things are changing are much longer compared to the time taken by light to cover distances equal to the length scales of the problem

Example:



An amplifier chip operating from 100 MHz to 10 GHz

100 MHz Operation
• Time scale of the problem = 1/(100 MHz) = 10 ns
• Length scale of the problem = 3 cm
• Time taken by light to travel 3 cm = 0.1 ns

Since 10 ns >> 0.1 ns, quasistatics is a valid means of analysis at 100 MHz

10 GHz Operation
• Time scale of the problem = 1/(10 GHz) = 0.1 ns
• Length scale of the problem = 3 cm
• Time taken by light to travel 3 cm = 0.1 ns

Quasistatics is not a valid means of analysis at 10 GHz

Electroquasistatics and Magnetoquasistatics - III

Question (contd..): How slowly is "slowly" ?

Electromagnetic wave frequency f and wavelength λ are related to the speed of the wave c by the relation:

$$f\,\lambda = c$$

Let: **L = length scale of the problem**

T = time scale of the problem ≈ 1/f

Condition for quasitatic analysis to be valid:

$$T >> \frac{L}{c}$$

$$\Rightarrow \quad cT >> L$$

$$\Rightarrow \quad \frac{c}{f} >> L$$

$$\Rightarrow \quad \lambda >> L$$

Quasistatic analysis is valid if the wavelength of electromagnetic wave at the frequency of interest is much longer than the length scales involved in the problem