## Lecture 19

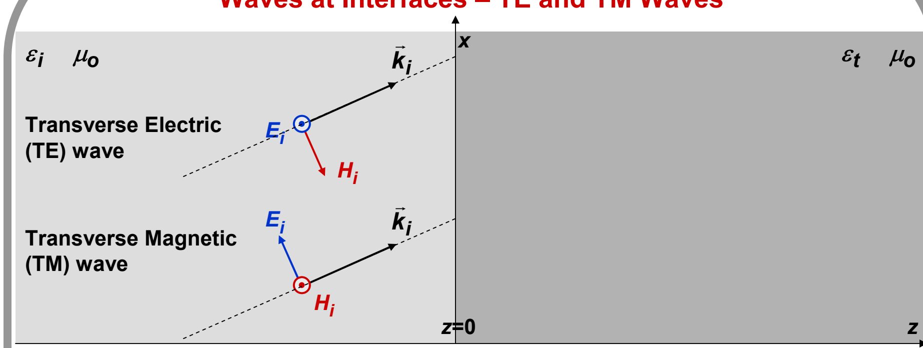
## Non-Normal Incidence of Waves at Interfaces

**In this lecture you will learn:**

- What happens when waves strike an interface between two different media coming at an angle
- Reflection and transmission of waves at interfaces
- Application of E-field and H-field boundary conditions
- Total internal reflection
- Brewster's angle



## Waves at Interfaces – TE and TM Waves

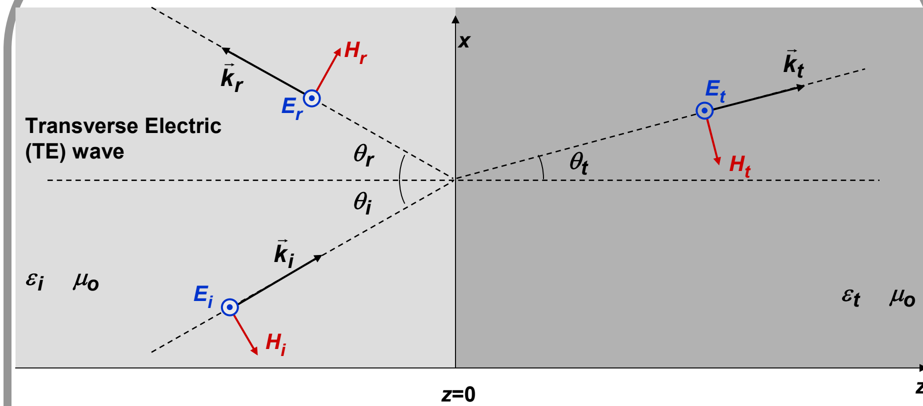



**Plane of Incidence:** The plane containing the incident wavevector $\vec{k}_i$ and a vector that is normal to the interface is called the plane of incidence (in the figure above the *x-z* plane is the plane of incidence)

**TE Wave:** If the E-field of the wave is perpendicular to the plane of incidence then the wave is called a TE-wave

**TM Wave:** If the H-field of the wave is perpendicular to the plane of incidence then the wave is called a TM-wave

## TE Wave - Wavevectors



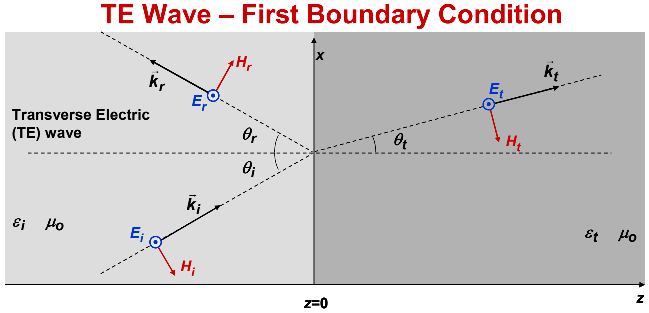

$$\vec{k}_i = k_{ix}\hat{x} + k_{iz}\hat{z} = k_i\,[\sin(\theta_i)\,\hat{x} + \cos(\theta_i)\,\hat{z}] \qquad k_i = \omega\sqrt{\mu_o\,\varepsilon_i} = \omega\frac{n_i}{c}$$

$$\vec{k}_r = k_{rx}\hat{x} + k_{rz}\hat{z} = k_r\,[\sin(\theta_r)\,\hat{x} - \cos(\theta_r)\,\hat{z}] \qquad k_r = k_i = \omega\sqrt{\mu_o\,\varepsilon_i} = \omega\frac{n_i}{c}$$

$$\vec{k}_t = k_{tx}\hat{x} + k_{tz}\hat{z} = k_t\,[\sin(\theta_t)\,\hat{x} + \cos(\theta_t)\,\hat{z}] \qquad k_t = \omega\sqrt{\mu_o\,\varepsilon_t} = \omega\frac{n_t}{c}$$



## TE Wave – First Boundary Condition




$$\vec{E}(\vec{r})\big|_{z<0} = \hat{y}\;E_i\;e^{-j\,\vec{k}_i\,.\,\vec{r}} + \hat{y}\;E_r\;e^{-j\,\vec{k}_r\,.\,\vec{r}}$$

$$\vec{E}(\vec{r})\big|_{z>0} = \hat{y}\;E_t\;e^{-j\,\vec{k}_t\,.\,\vec{r}}$$

$$\begin{cases} \vec{k}_i = k_i\,[\sin(\theta_i)\,\hat{x} + \cos(\theta_i)\,\hat{z}] \\ \vec{k}_r = k_r\,[\sin(\theta_r)\,\hat{x} - \cos(\theta_r)\,\hat{z}] \\ \vec{k}_t = k_t\,[\sin(\theta_t)\,\hat{x} + \cos(\theta_t)\,\hat{z}] \end{cases}$$

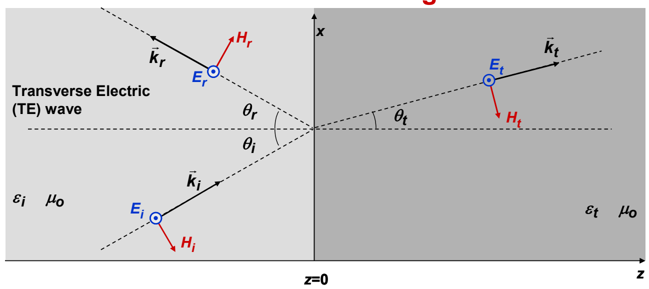
**Use boundary conditions:**

**(1) At $z = 0$ the E-field parallel to the interface must be continuous across the interface for all $x$**

This gives: $E_i\;e^{-j\,k_i\sin(\theta_i)x} + E_r\;e^{-j\,k_r\sin(\theta_r)x} = E_t\;e^{-j\,k_t\sin(\theta_t)x}$

ECE 303 – Fall 2005 – Farhan Rana – Cornell University

## TE Wave – Phase Matching Condition




$$E_i\, e^{-j k_i \sin(\theta_i) x} + E_r\, e^{-j k_r \sin(\theta_r) x} = E_t\, e^{-j k_t \sin(\theta_t) x}$$

The only way the above boundary condition can be satisfied for all $x$ is if all the $x$-dependent phase factors are the same (this is called "phase matching")

$$k_i \sin(\theta_i) = k_r \sin(\theta_r) = k_t \sin(\theta_t) \longrightarrow \left\{ k_{ix} = k_{rx} = k_{tx} \right.$$

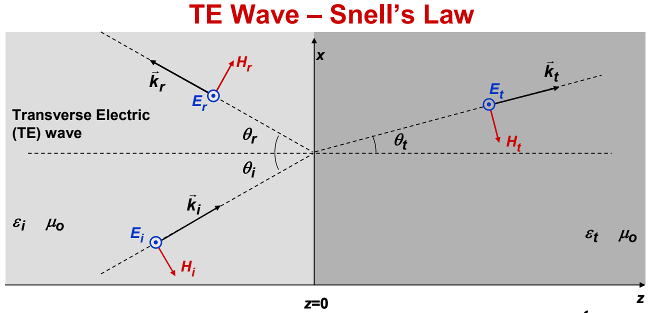
The first equality gives ( using $k_i = k_r$ ) :

$$\sin(\theta_i) = \sin(\theta_r) \quad \Rightarrow \quad \theta_i = \theta_r$$

angle of incidence equals the angle of reflection



## TE Wave – Snell's Law




$$k_i \sin(\theta_i) = k_r \sin(\theta_r) = k_t \sin(\theta_t) \longrightarrow \left\{ k_{ix} = k_{rx} = k_{tx} \right.$$

The second equality gives:

$$k_i \sin(\theta_i) = k_t \sin(\theta_t)$$

$$\Rightarrow \omega \frac{n_i}{c} \sin(\theta_i) = \omega \frac{n_t}{c} \sin(\theta_t)$$

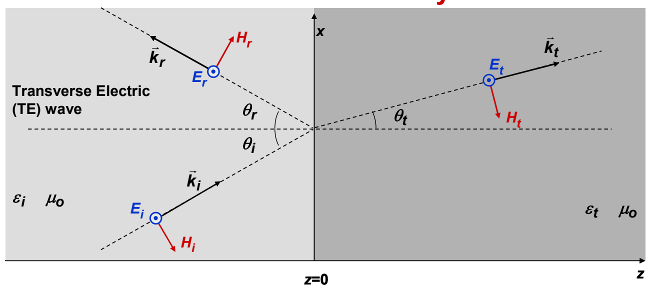
$$\Rightarrow n_i \sin(\theta_i) = n_t \sin(\theta_t)$$

Snell's Law

$$E_i\, e^{-j k_i \sin(\theta_i) x} + E_r\, e^{-j k_r \sin(\theta_r) x} = E_t\, e^{-j k_t \sin(\theta_t) x}$$

$$\Rightarrow \boxed{E_i + E_r = E_t} \qquad (1)$$

## TE Wave – Second Boundary Condition




**(2) At $z = 0$ the H-field component parallel to the interface must be continuous for all $x$**

$$\vec{H}(\vec{r})\Big|_{z<0} = \left(\hat{k}_i \times \hat{y}\right)\frac{E_i}{\eta_i}\, e^{-j\,\vec{k}_i\,.\,\vec{r}} + \left(\hat{k}_r \times \hat{y}\right)\frac{E_r}{\eta_i}\, e^{-j\,\vec{k}_r\,.\,\vec{r}}$$

$$\vec{H}(\vec{r})\Big|_{z>0} = \left(\hat{k}_t \times \hat{y}\right)\frac{E_t}{\eta_t}\, e^{-j\,\vec{k}_t\,.\,\vec{r}}$$

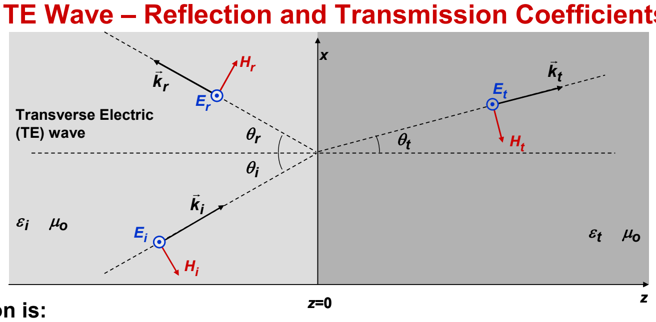
$$\begin{cases} \vec{k}_i = k_i\,[\sin(\theta_i)\,\hat{x} + \cos(\theta_i)\,\hat{z}] \\ \vec{k}_r = k_r\,[\sin(\theta_r)\,\hat{x} - \cos(\theta_r)\,\hat{z}] \\ \vec{k}_t = k_t\,[\sin(\theta_t)\,\hat{x} + \cos(\theta_t)\,\hat{z}] \end{cases}$$

$$-\hat{x}\cos(\theta_i)\frac{E_i}{\eta_i}\, e^{-j\,k_i\sin(\theta_i)x} + \hat{x}\cos(\theta_r)\frac{E_r}{\eta_i}\, e^{-j\,k_r\sin(\theta_r)x} = -\hat{x}\cos(\theta_t)\frac{E_t}{\eta_t}\, e^{-j\,k_t\sin(\theta_t)x}$$

$$\Rightarrow \boxed{\cos(\theta_i)\left[\frac{E_i}{\eta_i} - \frac{E_r}{\eta_i}\right] = \cos(\theta_t)\frac{E_t}{\eta_t}} \qquad \textbf{(2)}$$



## TE Wave – Reflection and Transmission Coefficients




**The solution is:**

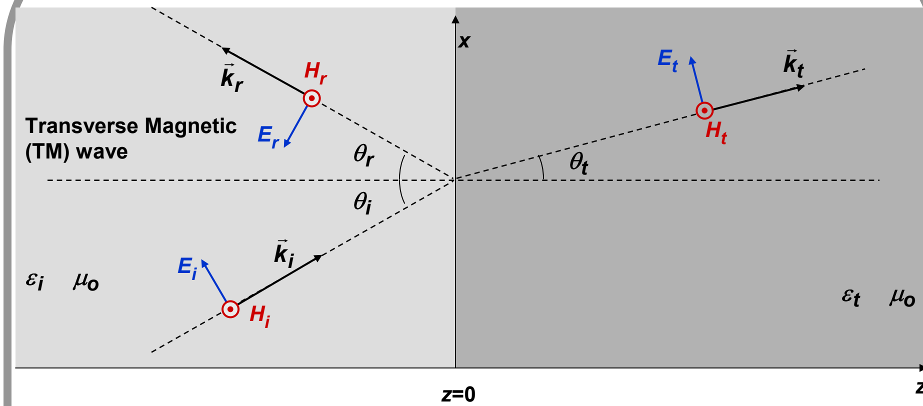
**Transmission coefficient**

$$T = \frac{E_t}{E_i} = \frac{2\,\dfrac{\eta_t/\cos(\theta_t)}{\eta_i/\cos(\theta_i)}}{\dfrac{\eta_t/\cos(\theta_t)}{\eta_i/\cos(\theta_i)} + 1} = \frac{2\,\dfrac{k_{iz}}{k_{tz}}}{\dfrac{k_{iz}}{k_{tz}} + 1} = \frac{2\,\dfrac{n_i/\cos(\theta_t)}{n_t/\cos(\theta_i)}}{\dfrac{n_i/\cos(\theta_t)}{n_t/\cos(\theta_i)} + 1}$$

**Reflection coefficient**

$$\Gamma = \frac{E_r}{E_i} = \frac{\dfrac{\eta_t/\cos(\theta_t)}{\eta_i/\cos(\theta_i)} - 1}{\dfrac{\eta_t/\cos(\theta_t)}{\eta_i/\cos(\theta_i)} + 1} = \frac{\dfrac{k_{iz}}{k_{tz}} - 1}{\dfrac{k_{iz}}{k_{tz}} + 1} = \frac{\dfrac{n_i/\cos(\theta_t)}{n_t/\cos(\theta_i)} - 1}{\dfrac{n_i/\cos(\theta_t)}{n_t/\cos(\theta_i)} + 1}$$

## TM Wave - Wavevectors




$$\vec{k}_i = k_{ix}\hat{x} + k_{iz}\hat{z} = k_i\left[\sin(\theta_i)\,\hat{x} + \cos(\theta_i)\,\hat{z}\right] \qquad k_i = \omega\sqrt{\mu_o\,\varepsilon_i} = \omega\frac{n_i}{c}$$

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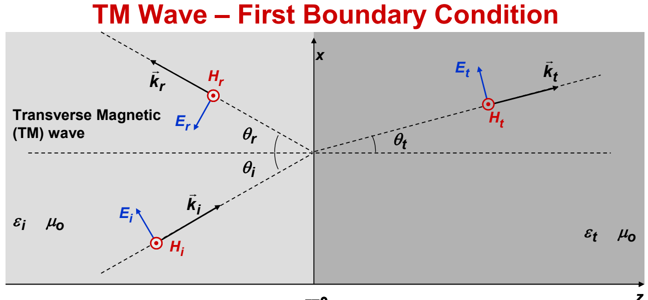
$$\vec{k}_t = k_{tx}\hat{x} + k_{tz}\hat{z} = k_t\left[\sin(\theta_t)\,\hat{x} + \cos(\theta_t)\,\hat{z}\right] \qquad k_t = \omega\sqrt{\mu_o\,\varepsilon_t} = \omega\frac{n_t}{c}$$



## TM Wave – First Boundary Condition




$$\vec{H}(\vec{r})\big|_{z<0} = \hat{y}\;H_i\;e^{-j\,\vec{k}_i\,.\,\vec{r}} + \hat{y}\;H_r\;e^{-j\,\vec{k}_r\,.\,\vec{r}}$$

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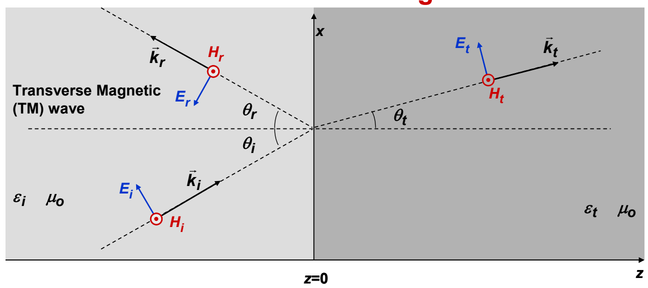
$$\begin{cases} \vec{k}_i = k_i\left[\sin(\theta_i)\,\hat{x} + \cos(\theta_i)\,\hat{z}\right] \\ \vec{k}_r = k_r\left[\sin(\theta_r)\,\hat{x} - \cos(\theta_r)\,\hat{z}\right] \\ \vec{k}_t = k_t\left[\sin(\theta_t)\,\hat{x} + \cos(\theta_t)\,\hat{z}\right] \end{cases}$$

**Use boundary conditions:**

**(1) At $z = 0$ the H-field parallel to the interface must be continuous across the interface for all $x$**

This gives: $H_i\;e^{-j\,k_i\sin(\theta_i)x} + H_r\;e^{-j\,k_r\sin(\theta_r)x} = H_t\;e^{-j\,k_t\sin(\theta_t)x}$

## TM Wave – Phase Matching Condition




$$H_i\, e^{-j k_i \sin(\theta_i) x} + H_r\, e^{-j k_r \sin(\theta_r) x} = H_t\, e^{-j k_t \sin(\theta_t) x}$$

The only way the above boundary condition can be satisfied for all $x$ is if all the $x$-dependent phase factors are the same (this is called "phase matching")

$$k_i \sin(\theta_i) = k_r \sin(\theta_r) = k_t \sin(\theta_t) \longrightarrow \left\{ k_{ix} = k_{rx} = k_{tx} \right.$$

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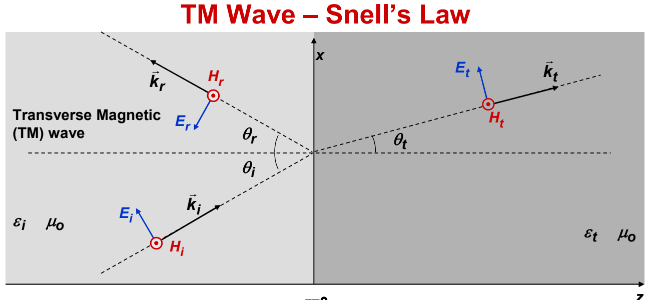
angle of incidence equals the angle of reflection



## TM Wave – Snell's Law




$$k_i \sin(\theta_i) = k_r \sin(\theta_r) = k_t \sin(\theta_t) \longrightarrow \left\{ k_{ix} = k_{rx} = k_{tx} \right.$$

The second equality gives:

$$k_i \sin(\theta_i) = k_t \sin(\theta_t)$$

$$\Rightarrow \omega \frac{n_i}{c} \sin(\theta_i) = \omega \frac{n_t}{c} \sin(\theta_t)$$

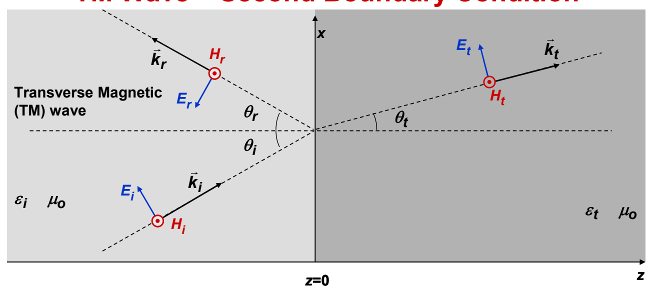
$$\Rightarrow n_i \sin(\theta_i) = n_t \sin(\theta_t)$$

Snell's Law

$$H_i\, e^{-j k_i \sin(\theta_i) x} + H_r\, e^{-j k_r \sin(\theta_r) x} = H_t\, e^{-j k_t \sin(\theta_t) x}$$

$$\Rightarrow \boxed{H_i + H_r = H_t} \qquad (1)$$

## TM Wave – Second Boundary Condition




(2) At $z = 0$ the E-field component parallel to the interface must be continuous for all $x$

$$\vec{E}(\vec{r})\Big|_{z<0} = -\left(\hat{k}_i \times \hat{y}\right)\eta_i\, H_i\, e^{-j\,\vec{k}_i\,.\,\vec{r}} - \left(\hat{k}_r \times \hat{y}\right)\eta_i\, H_r\, e^{-j\,\vec{k}_r\,.\,\vec{r}}$$

$$\vec{E}(\vec{r})\Big|_{z>0} = -\left(\hat{k}_t \times \hat{y}\right)\eta_t\, H_t\, e^{-j\,\vec{k}_t\,.\,\vec{r}}$$

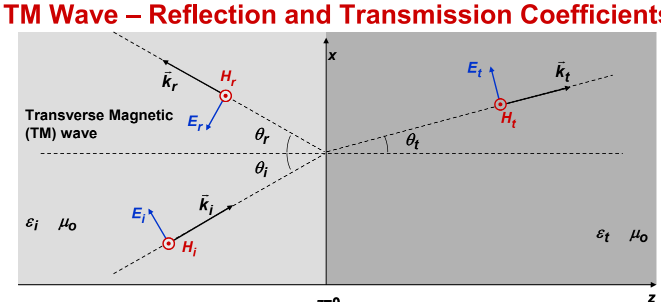
$$\begin{cases} \vec{k}_i = k_i\,[\sin(\theta_i)\,\hat{x} + \cos(\theta_i)\,\hat{z}] \\ \vec{k}_r = k_r\,[\sin(\theta_r)\,\hat{x} - \cos(\theta_r)\,\hat{z}] \\ \vec{k}_t = k_t\,[\sin(\theta_t)\,\hat{x} + \cos(\theta_t)\,\hat{z}] \end{cases}$$

$$\hat{x}\cos(\theta_i)\eta_i\, H_i\, e^{-j\,k_i \sin(\theta_i)x} - \hat{x}\cos(\theta_r)\eta_i\, H_r\, e^{-j\,k_r \sin(\theta_r)x} = \hat{x}\cos(\theta_t)\eta_t\, H_t\, e^{-j\,k_t \sin(\theta_t)x}$$

$$\Rightarrow \boxed{\cos(\theta_i)(\eta_i\, H_i - \eta_i\, H_r) = \cos(\theta_t)\eta_t\, H_t} \qquad (2)$$



## TM Wave – Reflection and Transmission Coefficients




The solution is:

Transmission coefficient

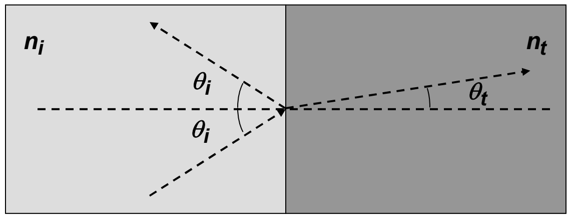
$$T^{TM} = \frac{H_t}{H_i} = \frac{2\,\dfrac{\eta_i/\cos(\theta_t)}{\eta_t/\cos(\theta_i)}}{\dfrac{\eta_i/\cos(\theta_t)}{\eta_t/\cos(\theta_i)} + 1} = \frac{2\,\dfrac{\varepsilon_t\, k_{iz}}{\varepsilon_i\, k_{tz}}}{\dfrac{\varepsilon_t\, k_{iz}}{\varepsilon_i\, k_{tz}} + 1} = \frac{2\,\dfrac{n_t/\cos(\theta_t)}{n_i/\cos(\theta_i)}}{\dfrac{n_t/\cos(\theta_t)}{n_i/\cos(\theta_i)} + 1}$$

Reflection coefficient

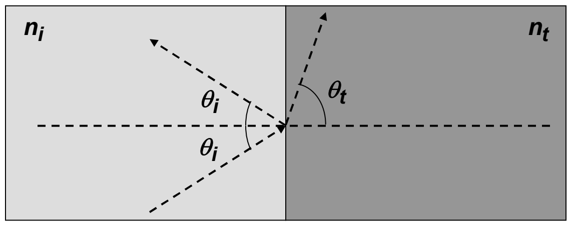
$$\Gamma^{TM} = \frac{H_r}{H_i} = \frac{\dfrac{\eta_i/\cos(\theta_t)}{\eta_t/\cos(\theta_i)} - 1}{\dfrac{\eta_i/\cos(\theta_t)}{\eta_t/\cos(\theta_i)} + 1} = \frac{\dfrac{\varepsilon_t\, k_{iz}}{\varepsilon_i\, k_{tz}} - 1}{\dfrac{\varepsilon_t\, k_{iz}}{\varepsilon_i\, k_{tz}} + 1} = \frac{\dfrac{n_t/\cos(\theta_t)}{n_i/\cos(\theta_i)} - 1}{\dfrac{n_t/\cos(\theta_t)}{n_i/\cos(\theta_i)} + 1}$$

## Snell's Law

$$n_i \sin(\theta_i) = n_t \sin(\theta_t)$$




- If $n_i < n_t$ then $\theta_t < \theta_i$ and the transmitted wave bends towards the normal

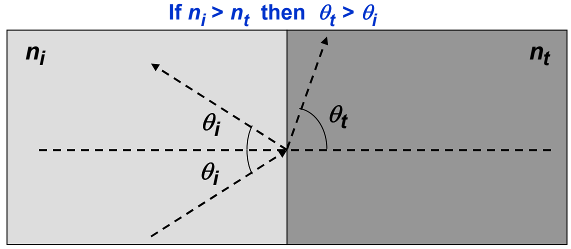



- If $n_i > n_t$ then $\theta_t > \theta_i$ and the transmitted wave bends away from the normal



## Total Internal Reflection – Critical Angle

If $n_i > n_t$ then $\theta_t > \theta_i$




If $\theta_i$ is increased, then $\theta_t$ will eventually become 90°

The value of $\theta_i$ for which $\theta_t$ is 90° is called the critical angle $\theta_c$

$$n_i \sin(\theta_i) = n_t \sin(\theta_t)$$

$$\Rightarrow \quad n_i \sin(\theta_c) = n_t \sin\left(\frac{\pi}{2}\right) \quad \Rightarrow \quad \boxed{\sin(\theta_c) = \frac{n_t}{n_i}}$$

What if $\theta_i$ is increased beyond $\theta_c$?

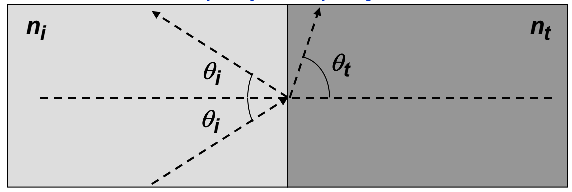
When $\theta_i$ is increased beyond $\theta_c$ the wave is not transmitted but is completely (100%) reflected at the interface back into the medium of incidence

This phenomenon is called total internal reflection – it happens for both TE and TM waves

## Total Internal Reflection – Phase Matching

$n_i > n_t$ and $\theta_i > \theta_c$

We need to consider in more detail what happens when the angle of incidence is greater than the critical angle




The phase matching condition gives us:

$$k_{ix} = k_{rx} = k_{tx}$$
$$\Rightarrow \quad k_{tx} = k_{ix} = k_i \sin(\theta_i)$$
$$\Rightarrow \quad k_{tx}^2 = k_i^2 \sin^2(\theta_i)$$
$$= \frac{\omega^2}{c^2} n_i^2 \sin^2(\theta_i)$$

We also know the dispersion relation for medium "*t*":

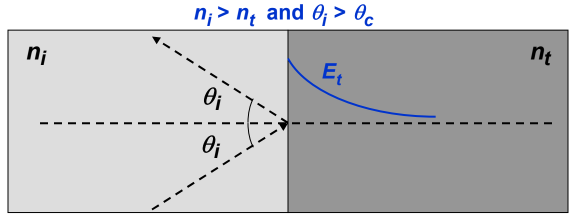
$$k_t = \frac{\omega}{c} n_t \quad \Rightarrow \quad k_t^2 = \frac{\omega^2}{c^2} n_t^2$$
$$\Rightarrow \quad k_{tx}^2 + k_{tz}^2 = \frac{\omega^2}{c^2} n_t^2$$



## Total Internal Reflection – Evanescent Wave

$n_i > n_t$ and $\theta_i > \theta_c$

If $\theta_i$ is larger than $\theta_c$ the wave in medium "*t*" is evanescent in the *z*-direction




The previous two equations imply: $k_{tz}^2 = \frac{\omega^2}{c^2} n_t^2 - k_{tx}^2 = \frac{\omega^2}{c^2}\left[n_t^2 - n_i^2 \sin^2(\theta_i)\right]$

$$\Rightarrow \quad k_{tz} = \frac{\omega}{c}\underbrace{\sqrt{n_t^2 - n_i^2 \sin^2(\theta_i)}}_{\text{-ve when } \theta_i > \theta_c}$$

For $\theta_i > \theta_c$ one can write:

$$k_{tz} = -j\frac{\omega}{c}\sqrt{n_i^2 \sin^2(\theta_i) - n_t^2} = -j\,k_{tz}''$$

→ The z-component of the wavevector has become completely imaginary

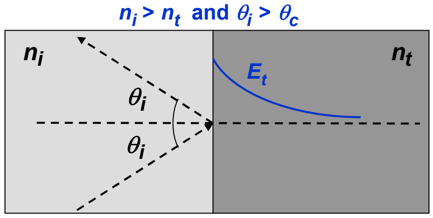
The E-field (assuming TE wave) in medium "*t*" is then:

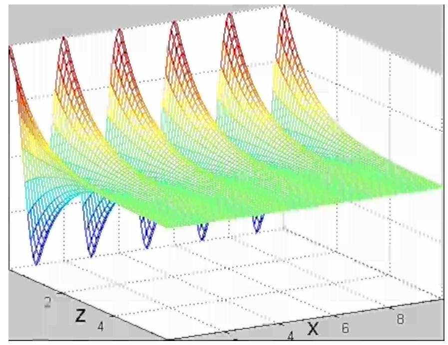
$$\vec{E}(\vec{r})\Big|_{z>0} = \hat{y}\, E_t\, e^{-j\,k_{tx}\,x}\, e^{-k_{tz}''\,z}$$

→ The field is evanescent in the *z*-direction in medium "*t*"

## Total Internal Reflection – Evanescent Wave

$n_i > n_t$ and $\theta_i > \theta_c$




The E-field (assuming TE wave) in medium "$t$" is:

$$\vec{E}(\vec{r})\Big|_{z>0} = \hat{y}\; E_t\; e^{-j\,k_{tx}\,x}\; e^{-k_{tz}''\,z} = \hat{y}\; |E_t|\; e^{j\phi}\; e^{-j\,k_{tx}\,x}\; e^{-k_{tz}''\,z}$$

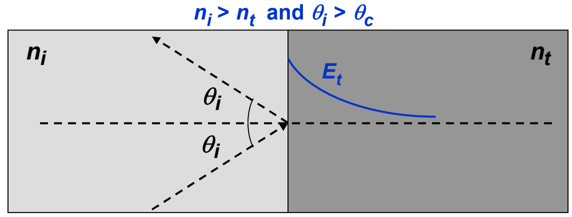
$$\Rightarrow \quad \vec{E}(\vec{r},t)\Big|_{z>0} = \hat{y}\; |E_t|\; e^{-k_{tz}''\,z}\; \cos(\omega t - k_{tx}\,x + \phi)$$

The wave is propagating along the interface (in the x-direction) but decaying (without spatial oscillations) in the z-direction



## Total Internal Reflection – Reflection Coefficient

$n_i > n_t$ and $\theta_i > \theta_c$




If $\theta_i$ is larger than $\theta_c$ the wave is completely reflected back into the medium of incidence

For $\theta_i > \theta_c$ one can write:

$$k_{tz} = -j\frac{\omega}{c}\sqrt{n_i^2 \sin^2(\theta_i) - n_t^2} = -j\,k_{tz}''$$

→ The z-component of the wavevector has become completely imaginary

The reflection coefficient for the E-field (assuming TE wave) is:

$$\Gamma = \frac{E_r}{E_i} = \frac{\dfrac{k_{iz}}{k_{tz}} - 1}{\dfrac{k_{iz}}{k_{tz}} + 1} = \frac{k_{iz} - k_{tz}}{k_{iz} + k_{tz}} = \frac{k_{iz} + jk_{tz}''}{k_{iz} - jk_{tz}''} = e^{j\varphi}$$

$$\Rightarrow \quad |\Gamma| = 1$$

The phase $\varphi$ of the reflection coefficient $\Gamma$ in total internal reflection is called the **Goos-Hanschen phase-shift**

## TE and TM Waves: Reflection Coefficients

**TE Waves**

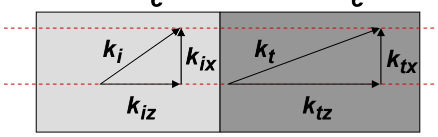
$$\Gamma = \frac{E_r}{E_i} = \frac{\dfrac{k_{iz}}{k_{tz}} - 1}{\dfrac{k_{iz}}{k_{tz}} + 1}$$

**TM Waves**

$$\Gamma^{TM} = \frac{H_r}{H_i} = \frac{\dfrac{\varepsilon_t k_{iz}}{\varepsilon_i k_{tz}} - 1}{\dfrac{\varepsilon_t k_{iz}}{\varepsilon_i k_{tz}} + 1}$$

**Question:** Can one ever get the reflection coefficient to go to zero (very desirable to get rid of unwanted reflections in optics)?

$$k_i = \omega \frac{n_i}{c} \qquad k_t = \omega \frac{n_t}{c}$$




$k_i \neq k_t \quad \left(\text{if } n_i \neq n_t\right)$

But $k_{ix} = k_{tx}$ (Phase matching)

$\Rightarrow \; k_{iz} \neq k_{tz}$

$\Rightarrow \; \Gamma$ is never zero for TE waves

**But one can have $\Gamma$=0 for TM waves if:**

$$\varepsilon_t k_{iz} = \varepsilon_i k_{tz}$$

$$\Rightarrow \; n_t \cos(\theta_i) = n_i \cos(\theta_t)$$

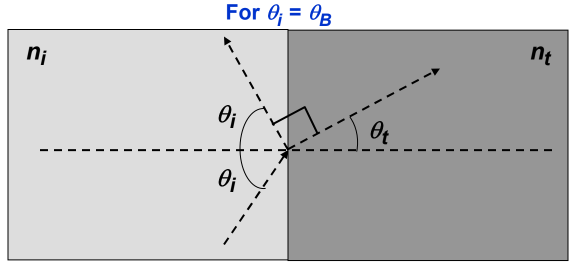


## TM Waves: Brewster's Angle

**One can have $\Gamma$=0 for TM waves if:**

$$n_t \cos(\theta_i) = n_i \cos(\theta_t)$$

For $\theta_i = \theta_B$




**Snell's law gives:**

$$n_i \sin(\theta_i) = n_t \sin(\theta_t)$$

**The above two equations will have a solution if and only if:**

$$\sin(\theta_i) = \cos(\theta_t) \quad \text{and} \quad \cos(\theta_i) = \sin(\theta_t)$$

**This happens when:** $\theta_i + \theta_t = \dfrac{\pi}{2}$

**The angle of incidence for which this happens is called the Brewster's angle $\theta_B$:**

$$\tan(\theta_i) = \frac{n_t}{n_i} \quad \Rightarrow \quad \theta_B = \tan^{-1}\left[\frac{n_t}{n_i}\right]$$