Lecture 17

Waves in Anisotropic Media

In this lecture you will learn:

- Wave propagation in anisotropic dielectric media
- Wave propagation in biaxial and uniaxial media
- Birefringence
- Quarter-wave and half-wave plates

Anisotropic Media

So far you have been dealing with materials that "looked" the same in all directions (i.e. isotropic)

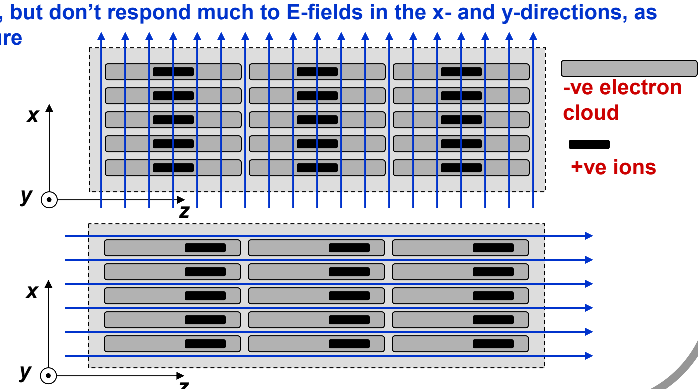
For isotropic media, the D-field is related to the E-field by one number – the dielectric permittivity:

$$\vec{D}(\vec{r}) = \varepsilon_o \vec{E}(\vec{r}) + \vec{P}(\vec{r}) = \varepsilon \vec{E}(\vec{r})$$

Now consider a material made up molecules that can easily be polarized by E-fields in the z-direction, but don't respond much to E-fields in the x- and y-directions, as shown in the figure

E-field in the x- or y-directions: material not much polarized

E-field in the z-direction: material strongly polarized



Dielectric Permittivity Tensor

In the most general case, the D-field is related to E-field through a dielectric permittivity tensor:

$$\vec{D}(\vec{r}) = \bar{\bar{\varepsilon}}.\vec{E}(\vec{r})$$

What this really means is that the x-, y-, and z-components of the D-field are related to the x-, y-, and z-components of the E-field by a permittivity matrix:

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

• This is the most general way of representing the effects of material polarization when the material is anisotropic

• The permittivity matrix is always symmetric, i.e. $\bar{\bar{\varepsilon}} = \left(\bar{\bar{\varepsilon}}\right)^T$. This follows from physical considerations that have to do with energy conservation and time-reversal symmetry. No media can violate this condition

Biaxial Media

The dielectric permittivity matrix is symmetric, i.e. $\bar{\bar{\varepsilon}} = \left(\bar{\bar{\varepsilon}}\right)^T$

There is a theorem in linear algebra that says that any symmetric matrix can be diagonalized by a suitable choice of the basis vectors (i.e. by a suitable choice for the orientation of the co-ordinate axes w.r.t. the material)

So in the most general case, if one chooses the orientation of the co-ordinate axes judiciously, the relation between the D-field and E-field becomes:

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

• This permittivity matrix is diagonal

• If the diagonal entries are all different the material is called biaxial

• The choice of co-ordinate axes that results in a diagonal permittivity matrix is called the principal axes of the material

Uniaxial Media

In a uniaxial media, two of the diagonal entries of the permittivity matrix are the same and one is different, for example:

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

- If the E-field is only along x-axis:

$$D_x(\vec{r}) = \varepsilon^e \, E_x(\vec{r})$$

- If the E-field is only along y-axis:

$$D_y(\vec{r}) = \varepsilon^o \, E_y(\vec{r})$$

- If the E-field is only along z-axis:

$$D_z(\vec{r}) = \varepsilon^o \, E_z(\vec{r})$$

For the case shown above, x-axis is called the extraordinary axis, and y-axis and z-axis are called the ordinary axes

Many important materials are uniaxial (e.g. calcite, mica, quartz, etc)

Wave Propagation in Anisotropic Media - I

Faraday's Law:

$$\nabla \times \vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\vec{H}(\vec{r})$$

Ampere's Law:

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\,\omega\,\vec{D}(\vec{r}) \quad (\vec{J}(\vec{r}) \to 0)$$

For anisotropic media:

$$\vec{D}(\vec{r}) = \overline{\overline{\varepsilon}}\,.\vec{E}(\vec{r})$$

Wave equation:

$$\nabla \times \nabla \times \vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\nabla \times \vec{H}(\vec{r}) = \omega^2\,\mu_o\,\vec{D}(\vec{r})$$

$$\Rightarrow \quad \nabla\left(\nabla.\vec{E}(\vec{r})\right) - \nabla^2\,\vec{E}(\vec{r}) = \omega^2\,\mu_o\,\vec{D}(\vec{r}) \quad (\nabla.\vec{E}(\vec{r}) \to 0)$$

$$\Rightarrow \quad \nabla^2\,\vec{E}(\vec{r}) = -\omega^2\,\mu_o\,\vec{D}(\vec{r})$$

Wait a minute !! Gauss' Law states that divergence of D-field (not E-field) is zero (if there is no charge density):

$$\nabla.\vec{D}(\vec{r}) = \nabla.\left(\overline{\overline{\varepsilon}}.\vec{E}(\vec{r})\right) = 0$$

Its not obvious that the above implies:

$$\nabla.\vec{E}(\vec{r}) = 0$$

But $\nabla.\vec{E}(\vec{r}) = 0$ turns out to be true for plane waves propagating in anisotropic media

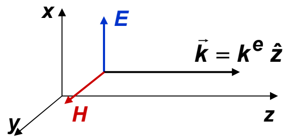
Wave Propagation in Uniaxial Media - I

The starting point is the following wave equation:

$$\nabla^2 \vec{E}(\vec{r}) = -\omega^2 \mu_o D(\vec{r})$$

Consider the following uniaxial media:

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$



And a plane wave polarized in the x-direction and traveling in +z-direction

$$\vec{E}(\vec{r}) = \hat{x}\ E_o\ e^{-jkz}$$

Substitute in the wave equation and use the permittivity tensor, to get:

$$\nabla^2 \vec{E}(\vec{r}) = -\omega^2 \mu_o D(\vec{r})$$

$$\Rightarrow \quad -k^2 \left(\hat{x} E_o\ e^{-jkz}\right) = -\omega^2 \mu_o \varepsilon^e \left(\hat{x} E_o\ e^{-jkz}\right)$$

$$\Rightarrow \quad k = \omega\sqrt{\mu_o \varepsilon^e} = k^e$$

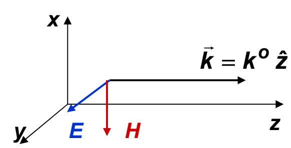
So a plane wave polarized in the x-direction and traveling in +z-direction is given as:

$$\vec{E}(\vec{r}) = \hat{x}\ E_o\ e^{-jk^e z}$$

Wave Propagation in Uniaxial Media - II

Again consider the same uniaxial media:

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$



And a plane wave polarized in the y-direction and traveling in +z-direction:

$$\vec{E}(\vec{r}) = \hat{y}\ E_o\ e^{-jkz}$$

Substitute in the wave equation and use the permittivity tensor, to get:

$$\nabla^2 \vec{E}(\vec{r}) = -\omega^2 \mu_o D(\vec{r})$$

$$\Rightarrow \quad -k^2 \left(\hat{y} E_o\ e^{-jkz}\right) = -\omega^2 \mu_o \varepsilon^o \left(\hat{y} E_o\ e^{-jkz}\right)$$

$$\Rightarrow \quad k = \omega\sqrt{\mu_o \varepsilon^o} = k^o$$

So a plane wave polarized in the y-direction and traveling in +z-direction is given as:

$$\vec{E}(\vec{r}) = \hat{y}\ E_o\ e^{-jk^o z}$$

Wave Propagation in Uniaxial Media - III

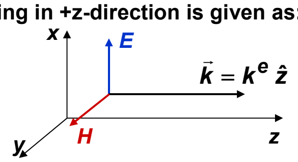
So for the uniaxial media in which the permittivity tensor is:

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

A plane wave polarized in the x-direction and traveling in +z-direction is given as:

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k^e z}$$

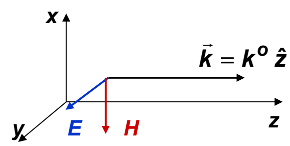
$$k^e = \omega\sqrt{\mu_o\, \varepsilon^e} = \omega \frac{n^e}{c}$$



A plane wave polarized in the y-direction and traveling in +z-direction is given as:

$$\vec{E}(\vec{r}) = \hat{y}\, E_o\, e^{-j k^o z}$$

$$k^o = \omega\sqrt{\mu_o\, \varepsilon^o} = \omega \frac{n^o}{c}$$



Waves polarized in different directions (but traveling in the same direction) have different wavevectors

Wave Propagation in Uniaxial Media - IV

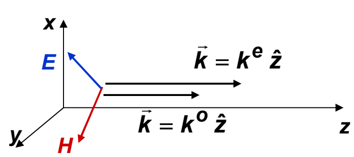
Consider a uniaxial media in which the permittivity tensor is:

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

Question: How does a plane wave with both x- and y-components propagate?

Answer: Each component propagates with its own wavevector (think linear superposition)

$$\vec{E}(\vec{r}) = \hat{x}\, E_x\, e^{-j k^e z} + \hat{y}\, E_y\, e^{-j k^o z}$$

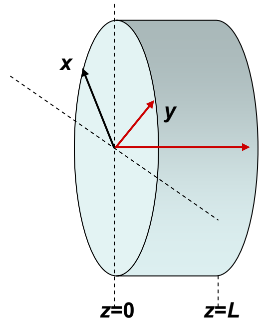


The above expression will satisfy the wave equation: $\nabla^2 \vec{E}(\vec{r}) = -\omega^2\, \mu_o\, D(\vec{r})$

- The plane wave has no single wavevector.
- The plane wave has two wavevectors – one associated with the x-component of the E-field and one associated with the y-component of the E-field

Uniaxial Media and Birefringence

Consider a slab of uniaxial media with principal axes as shown:



$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

$$k^e = \omega\sqrt{\mu_o\,\varepsilon^e} = \omega\frac{n^e}{c}$$

$$k^o = \omega\sqrt{\mu_o\,\varepsilon^o} = \omega\frac{n^o}{c}$$

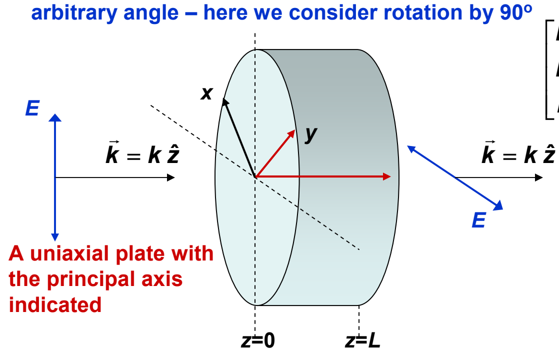
- If $k^e < k^o$ then the extraordinary axis are called the fast axis and the ordinary axis are called the slow axis (because the wave travels faster if it is polarized along the fast axis)
- If $k^e > k^o$ then the extraordinary axis are called the slow axis and the ordinary axis are called the fast axis

- The phenomenon where waves of different polarization direction travel at different velocities is called birefringence

Uniaxial Media Applications – Half-Wave Plates

Half-wave plates are used to rotate the polarization of a plane wave by an arbitrary angle – here we consider rotation by 90°

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$



$$k^e = \omega\sqrt{\mu_o\,\varepsilon^e} = \omega\frac{n^e}{c}$$

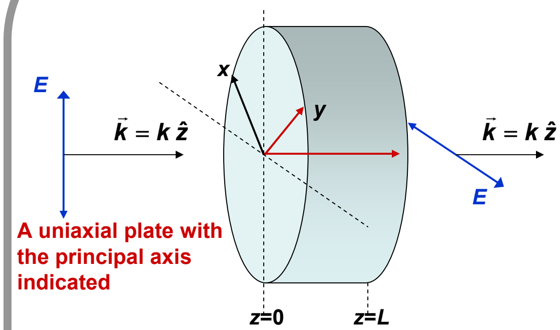
$$k^o = \omega\sqrt{\mu_o\,\varepsilon^o} = \omega\frac{n^o}{c}$$

At z=0, the field inside the plate is: $\left.\vec{E}(\vec{r})\right|_{z=0} = E_o\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)$

Then the field inside the plate for any z is: $\vec{E}(\vec{r}) = \frac{E_o}{\sqrt{2}}\left(\hat{x}\,e^{-j\,k^e z} + \hat{y}\,e^{-j\,k^o z}\right)$

The field inside the plate for $z = L$ is: $\left.\vec{E}(\vec{r})\right|_{z=L} = \frac{E_o}{\sqrt{2}}\left(\hat{x}\,e^{-j\,k^e L} + \hat{y}\,e^{-j\,k^o L}\right)$

Uniaxial Media Applications – Half-Wave Plates



$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

$$k^e = \omega\sqrt{\mu_o\,\varepsilon^e} = \omega\frac{n^e}{c}$$

$$k^o = \omega\sqrt{\mu_o\,\varepsilon^o} = \omega\frac{n^o}{c}$$

The field inside the plate for $z = L$ is: $\vec{E}(\vec{r})\Big|_{z=L} = \frac{E_o}{\sqrt{2}}\, e^{-j\,k^e L}\left(\hat{x} + \hat{y}\, e^{-j\left(k^o - k^e\right)L}\right)$

If: $\left(k^o - k^e\right)L = (2m+1)\pi \qquad m = 0, \pm 1, \pm 2, \ldots\ldots$

Then the field inside the plate for $z = L$ is: $\vec{E}(\vec{r})\Big|_{z=L} = E_o\; e^{-j\,k^e L}\left(\frac{\hat{x} - \hat{y}}{\sqrt{2}}\right)$

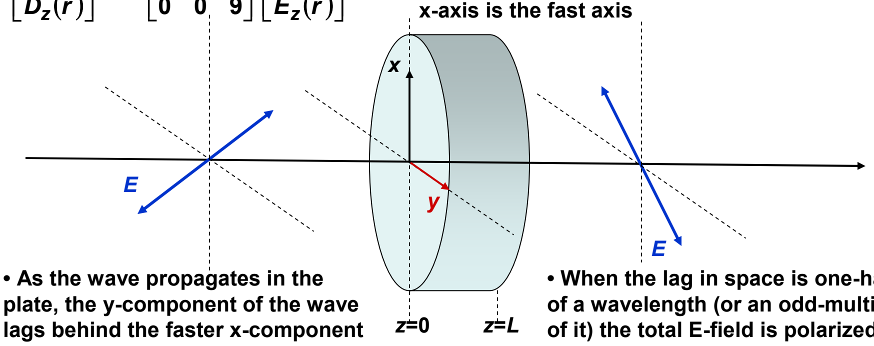
Output polarization in direction orthogonal to the incident polarization
⇒ Polarization rotated through 90-degrees

Half-Wave Plates: An Example

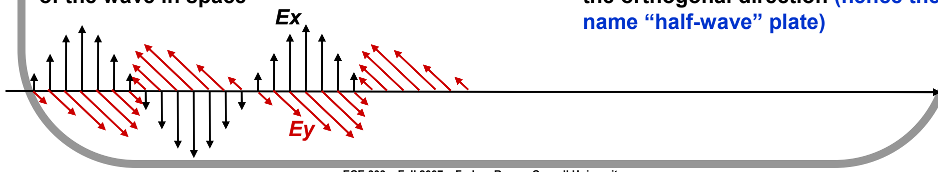
$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \varepsilon_o \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

$$k^e = \omega\sqrt{\mu_o\,\varepsilon^e} = \omega\frac{2}{c} \qquad k^o = \omega\sqrt{\mu_o\,\varepsilon^o} = \omega\frac{3}{c}$$

y-axis and z-axis are the slow axes
x-axis is the fast axis

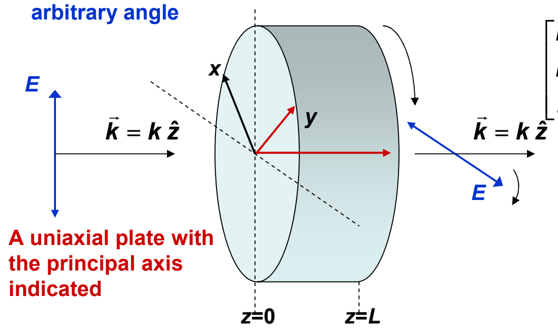


- As the wave propagates in the plate, the y-component of the wave lags behind the faster x-component of the wave in space
- When the lag in space is one-half of a wavelength (or an odd-multiple of it) the total E-field is polarized in the orthogonal direction (hence the name "half-wave" plate)



Uniaxial Media Applications – Half-Wave Plates

Half-wave plates are used to rotate the polarization of a plane wave by an arbitrary angle



$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

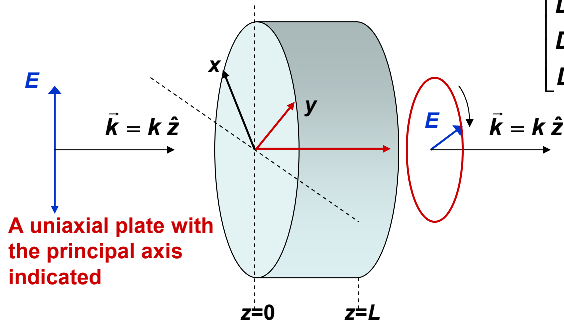
$$k^e = \omega\sqrt{\mu_o\,\varepsilon^e} = \omega\frac{n^e}{c}$$

$$k^o = \omega\sqrt{\mu_o\,\varepsilon^o} = \omega\frac{n^o}{c}$$

- The output polarization can be rotated through any angle by rotating the axis of the plate w.r.t. the incident polarization

Uniaxial Media Applications – Quarter-Wave Plates

Quarter-wave plates are used to turn linearly polarized waves into circularly or elliptically polarized waves and vice versa



$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

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The field inside the plate for any z is: $\vec{E}(\vec{r}) = \frac{E_o}{\sqrt{2}}\left(\hat{x}\,e^{-j\,k^e z} + \hat{y}\,e^{-j\,k^o z}\right)$

The field inside the plate for $z = L$ is: $\left.\vec{E}(\vec{r})\right|_{z=L} = \frac{E_o}{\sqrt{2}}\left(\hat{x}\,e^{-j\,k^e L} + \hat{y}\,e^{-j\,k^o L}\right)$

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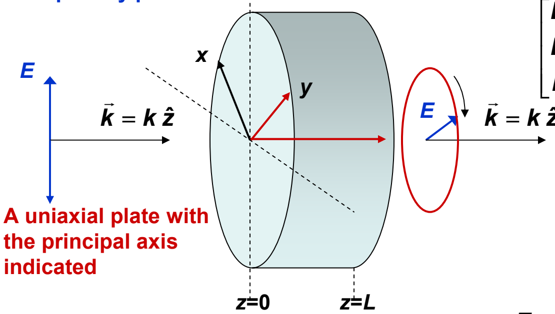

A uniaxial plate with the principal axis indicated

$$\begin{bmatrix} D_x(\vec{r}) \\ D_y(\vec{r}) \\ D_z(\vec{r}) \end{bmatrix} = \begin{bmatrix} \varepsilon^e & 0 & 0 \\ 0 & \varepsilon^o & 0 \\ 0 & 0 & \varepsilon^o \end{bmatrix} \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

$$k^e = \omega\sqrt{\mu_o\,\varepsilon^e} = \omega\frac{n^e}{c}$$

$$k^o = \omega\sqrt{\mu_o\,\varepsilon^o} = \omega\frac{n^o}{c}$$

The field inside the plate for $z = L$ is: $\left.\vec{E}(\vec{r})\right|_{z=L} = \frac{E_o}{\sqrt{2}}\, e^{-j\,k^e L}\left(\hat{x} + \hat{y}\, e^{-j\left(k^o - k^e\right)L}\right)$

If: $\left(k^o - k^e\right)L = (2m+1)\frac{\pi}{2} \qquad m = 0, \pm 1, \pm 2, \ldots\ldots$

Then the field inside the plate for $z = L$ is: $\left.\vec{E}(\vec{r})\right|_{z=L} = E_o\; e^{-j\,k^e L}\left(\frac{\hat{x} \pm j\,\hat{y}}{\sqrt{2}}\right)$

Output polarization is either left-hand or right-hand circular – depending upon the value of *m* (i.e. the thickness L of the plate)

Prelim 1 Results