Lecture 16 (b)

Waves in Isotropic Media: Plasmas and Dispersive Media

In this lecture you will learn:

- **Wave propagation in plasmas**
- **Wave propagation in dispersive media**
- **Phase and group velocities**

Plasmas

What is a Plasma?

A plasma is an assembly of positive and negative charged particles with a net zero time-average charge density

Examples of Plasmas:

1) Gases in which the electrons have been stripped off the atoms – resulting in a mixture of positive ions and electrons

Examples:

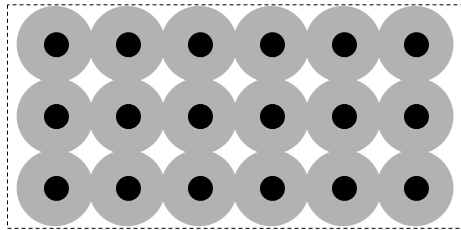
a) Surface of the Sun
b) Hydrogen ions and electrons in a fusion reactor
c) Earth's Ionosphere

2) Atoms and electrons making up solids (semiconductors, metals, etc) can also be described as a plasma - although in this case the positive charges are fixed

Plasmas

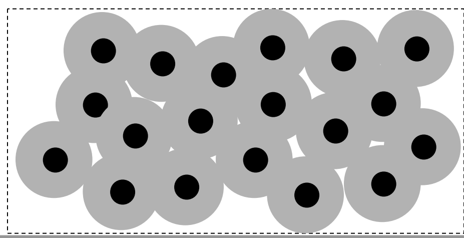
+ve ions

-ve electron cloud



A solid state plasma

- In plasmas, electrons are not attached to any one particular positive ion (or atom) but move about freely
- On average (spatial and temporal average), plasmas are charge neutral (i.e. electrons tend to spend more time in the vicinity of the positive ions (or atoms) than away from them – hence the pictures shown



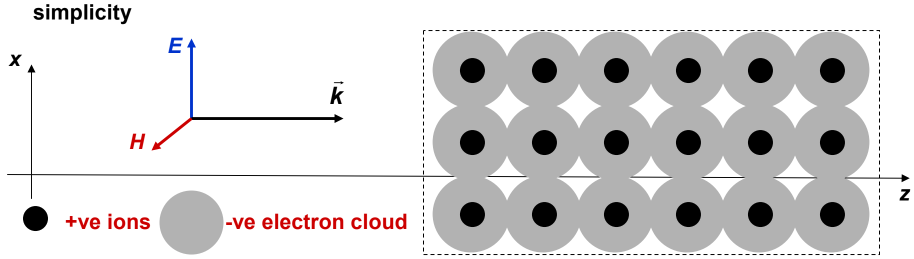
A gaseous plasma

Waves in Plasmas – Electron Dynamics

What happens when a plasma is subjected to a time-dependent E-field (like that of a plane wave)?

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-j k z}$$

In most plasmas, the positive ions (or atoms) are much heavier than the negative electrons and one may ignore the motion of the ions in response to an E-field for simplicity



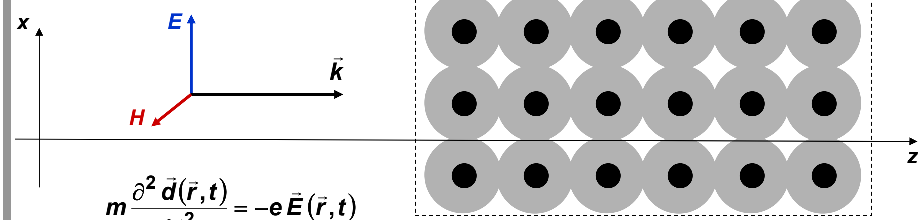
Let the displacement of the electrons at the location $\vec{r}$ from their average position be given by the vector $\vec{d}(\vec{r},t)$

Newton's second law gives:

$$m \frac{\partial^2 \vec{d}(\vec{r},t)}{\partial t^2} = \vec{F}(\vec{r},t) = -e\,\vec{E}(\vec{r},t)$$

Waves in Plasmas – Material Polarization

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-jkz}$$



$$m \frac{\partial^2 \vec{d}(\vec{r},t)}{\partial t^2} = -e\,\vec{E}(\vec{r},t)$$

Use phasors to solve the differential equation:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{ \vec{E}(\vec{r})\, e^{j\omega t} \right\} \qquad \vec{d}(\vec{r},t) = \mathrm{Re}\left\{ \vec{d}(\vec{r})\, e^{j\omega t} \right\}$$

To get:

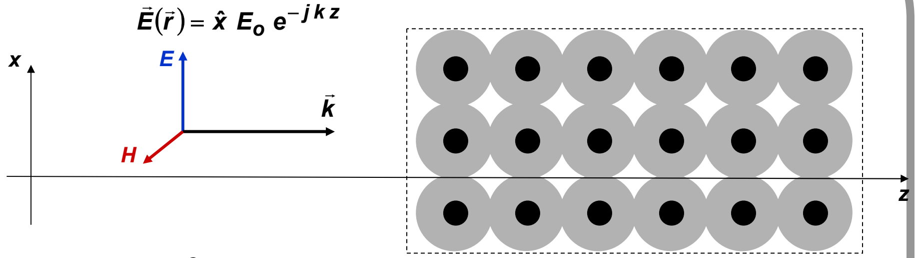
$$\Rightarrow\; -m\omega^2\, \vec{d}(\vec{r}) = -e\,\vec{E}(\vec{r})$$

$$\Rightarrow\; \vec{d}(\vec{r}) = \frac{e}{m\omega^2}\vec{E}(\vec{r})$$

Dipole moment phasor $= \vec{p}(\vec{r}) = -e\,\vec{d}(\vec{r})$

Waves in Plasmas – Dielectric Permittivity

$$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-jkz}$$



$$\Rightarrow\; \vec{d}(\vec{r}) = \frac{e}{m\omega^2}\vec{E}(\vec{r})$$

Dipole moment phasor $= \vec{p}(\vec{r}) = -e\,\vec{d}(\vec{r})$

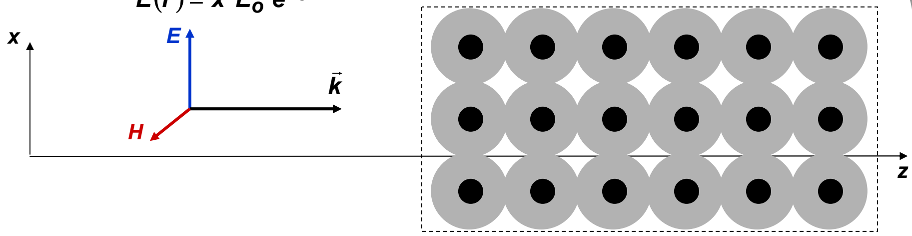
Material polarization phasor $= \vec{P}(\vec{r}) = N\vec{p}(\vec{r}) = N\left[-e\,\vec{d}(\vec{r}) \right] = -\frac{Ne^2}{m\omega^2}\vec{E}(\vec{r})$

Dielectric permittivity:

$$\vec{D}(\vec{r}) = \varepsilon_o \vec{E}(\vec{r}) + \vec{P}(\vec{r}) = \varepsilon_o \left[1 - \frac{Ne^2}{m\varepsilon_o\omega^2} \right] \vec{E}(\vec{r}) = \varepsilon(\omega)\vec{E}(\vec{r})$$

Waves in Plasmas – Dispersion Relation

$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-jkz}$



Dielectric permittivity

$$\vec{D}(\vec{r}) = \varepsilon_o \vec{E}(\vec{r}) + \vec{P}(\vec{r}) = \varepsilon_o\left[1 - \frac{N e^2}{m\,\varepsilon_o\,\omega^2}\right]\vec{E}(\vec{r}) = \varepsilon(\omega)\vec{E}(\vec{r})$$

$$\varepsilon(\omega) = \varepsilon_o\left[1 - \frac{N e^2}{m\,\varepsilon_o\,\omega^2}\right] = \varepsilon_o\left[1 - \frac{\omega_p^2}{\omega^2}\right] \longrightarrow \left\{ \omega_p = \sqrt{\frac{N e^2}{m\,\varepsilon_o}} \right.$$ plasma frequency

A plane wave will satisfy the complex wave equation: $\nabla^2 \vec{E}(\vec{r}) = -\omega^2 \mu_o\, \varepsilon(\omega) \vec{E}(\vec{r})$

Dispersion relation: $k = \omega\sqrt{\mu_o\, \varepsilon(\omega)} \quad \Rightarrow \quad k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$

Waves in Plasmas – Wave Propagation and Evanescent Waves

Plane wave: $\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-jkz}$

Dispersion relation: $k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$

$$\left\{ \begin{array}{l} \varepsilon(\omega) = \varepsilon_o\left[1 - \frac{\omega_p^2}{\omega^2}\right] \\ \eta(\omega) = \sqrt{\frac{\mu_o}{\varepsilon(\omega)}} \end{array} \right.$$

CASE 1 ($\omega > \omega_p$)

In this case one has normal wave propagation just as if the plasma was a dielectric medium with refractive index n given by:

$$n = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_o}} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

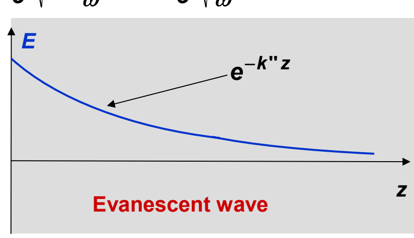
CASE 2 ($\omega < \omega_p$)

The dispersion relation becomes: $k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} = -j\frac{\omega}{c}\sqrt{\frac{\omega_p^2}{\omega^2} - 1} = -jk'' \qquad (k' = 0)$

Wavevector is completely imaginary !

Plane wave decays exponentially with distance (without any spatial oscillations)

$\vec{E}(\vec{r}) = \hat{x}\, E_o\, e^{-jkz} = \hat{x}\, E_o\, e^{-k''z}$

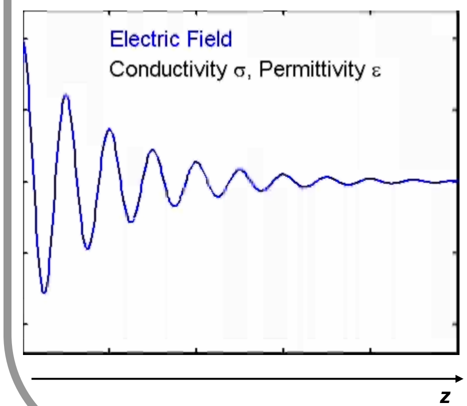


Decaying Waves Vs Evanescent Waves

Decaying wave in a lossy/conductive medium

$$\vec{E}(\vec{r}) = \hat{x}\ E_o\ e^{-jk'z}\ e^{-k''z}$$

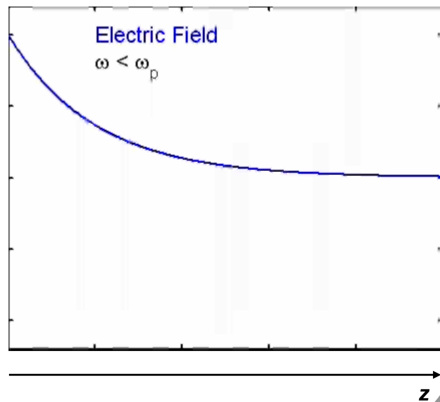
$$\Rightarrow \vec{E}(\vec{r},t) = \hat{x}\ E_o\ e^{-k''z} \cos(\omega t - k'z)$$



Evanescent wave in a plasma ($\omega < \omega_p$)

$$\vec{E}(\vec{r}) = \hat{x}\ E_o\ e^{-k''z}$$

$$\Rightarrow \vec{E}(\vec{r},t) = \hat{x}\ E_o\ e^{-k''z} \cos(\omega t)$$



Waves in Plasmas – Power Flow for Evanescent Waves

Evanescent waves ($\omega < \omega_p$):

$$\left\{ \frac{1}{\eta(\omega)} = \sqrt{\frac{\mu_o}{\varepsilon(\omega)}} = -j\,\frac{k''}{\mu_o\ \omega} \right.$$

Evanescent wave:

$$\vec{E}(\vec{r}) = \hat{x}\ E_o\ e^{-k''z} \quad \Rightarrow \quad \vec{H}(\vec{r}) = \hat{y}\,\frac{E_o}{\eta(\omega)}\,e^{-k''z} = -j\,\frac{k''}{\mu_o\ \omega}\,\hat{y}\,E_o\ e^{-k''z}$$

H-field is 90-degrees out of phase with the E-field

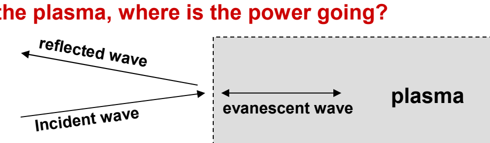
Poynting vector and time average power per unit area:

$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right] = \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r}) \times \vec{H}^*(\vec{r})\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\hat{z}\,\frac{E_o^2}{(\eta(\omega))^*}\ e^{-2k''z}\right] = 0 \longrightarrow \left\{ \text{No power is carried by the evanescent wave} \right.$$

So if power is not traveling into the plasma, where is the power going?

When $\omega < \omega_p$ all power in the Incident wave goes into the reflected wave at the surface of the plasma



Wave Propagation in Dispersive Media

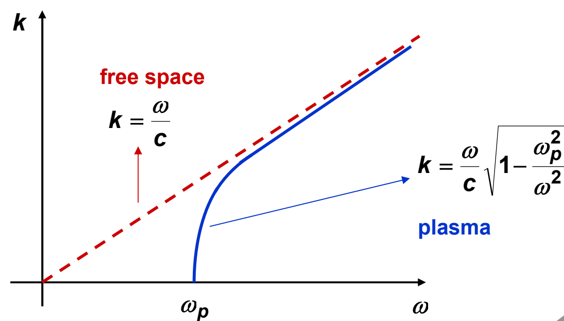
Any medium for which the permittivity is a function of frequency is called a **dispersive** medium

Example:

Plasmas: $\varepsilon(\omega) = \varepsilon_o \left[1 - \frac{\omega_p^2}{\omega^2}\right]$ $\quad k = \omega\sqrt{\mu_o\,\varepsilon(\omega)} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$

You will see more examples of dispersive media later in the course

The k-vs-ω relations (or the dispersion relations) for a medium are usually plotted as follows



Wave Packets - I

• Electromagnetic wave signals are transmitted not as plane waves of a particular frequency :

$$\vec{E}(z) = \hat{x}\,E_o\,e^{-jkz}$$

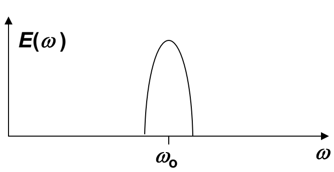
or

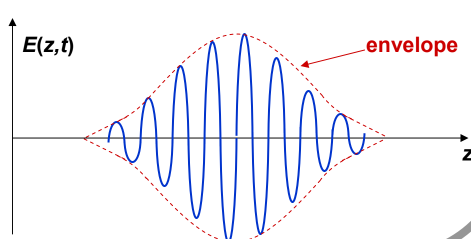
$$\vec{E}(z,t) = \text{Re}\left[\hat{x}\,E_o\,e^{j(\omega t - kz)}\right]$$

that extend in the z-direction from –ve infinity to +ve infinity, but in the form of **wave-packets** that are somewhat localized in space

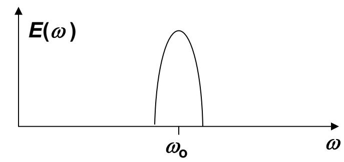
• A **wave-packet** is a linear superposition of plane waves of different frequencies:

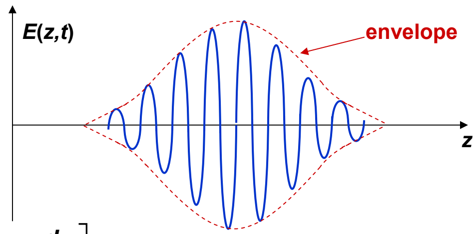
$$\vec{E}(z,t) = \text{Re}\left[\int_0^\infty \hat{x}\,E(\omega)\,e^{j(\omega t - k(\omega) z)}\,\frac{d\omega}{2\pi}\right]$$





Wave Packets - II





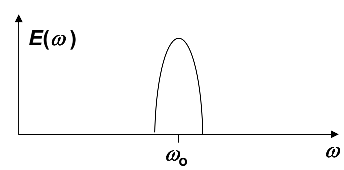
$$\vec{E}(z,t) = \text{Re}\left[\int_0^{\infty} \hat{x}\; E(\omega)\; e^{j(\omega t - k(\omega) z)}\; \frac{d\omega}{2\pi}\right]$$

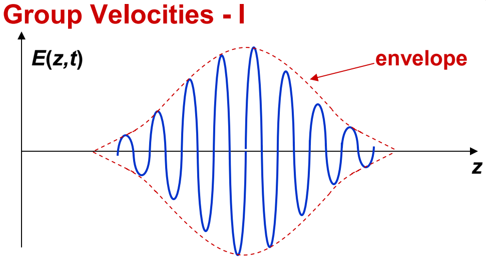
Let: $\omega = \omega_o + \Delta\omega$ ⟶ **Change of variables**

$$k(\omega + \Delta\omega) = k(\omega_o) + \left.\frac{dk}{d\omega}\right|_{\omega=\omega_o} \Delta\omega$$ ⟶ **Taylor expansion**

$$\Rightarrow \quad \vec{E}(z,t) = \text{Re}\left[\int_0^{\infty} \hat{x}\; E(\omega_o + \Delta\omega)\; e^{j\omega_o\left(t - \frac{k(\omega_o)}{\omega_o} z\right) + j\Delta\omega\left(t - \frac{dk}{d\omega} z\right)}\; \frac{d\,\Delta\omega}{2\pi}\right]$$

Phase and Group Velocities - I





$$\Rightarrow \quad \vec{E}(z,t) = \text{Re}\left[\int_0^{\infty} \hat{x}\; E(\omega_o + \Delta\omega)\; e^{j\omega_o\left(t - \frac{k(\omega_o)}{\omega_o} z\right) + j\Delta\omega\left(t - \frac{dk}{d\omega} z\right)}\; \frac{d\,\Delta\omega}{2\pi}\right]$$

Define **phase velocity** v_p as: $v_p = \left.\frac{\omega}{k}\right|_{\omega=\omega_o}$

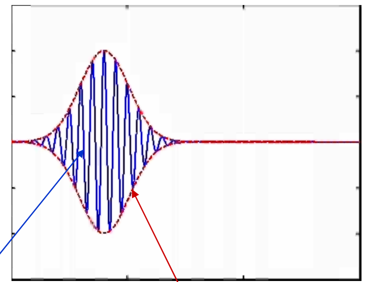
Define **group velocity** v_g as: $v_g = \left.\frac{d\omega}{dk}\right|_{\omega=\omega_o}$

$$\Rightarrow \quad \vec{E}(z,t) = \text{Re}\left[e^{-j\frac{\omega_o}{v_p}(z - v_p t)} \int_0^{\infty} \hat{x}\; E(\omega_o + \Delta\omega)\; e^{-j\frac{\Delta\omega}{v_g}(z - v_g t)}\; \frac{d\,\Delta\omega}{2\pi}\right]$$

Phase and Group Velocities - II

Phase velocity $= v_p = \dfrac{\omega}{k}$

Group velocity $= v_g = \dfrac{d\omega}{dk}$



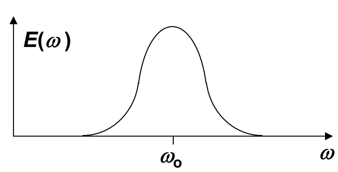
$$\vec{E}(z,t) = \mathrm{Re}\left[e^{-j\frac{\omega_o}{v_p}\left(z - v_p t\right)} \int_0^{\infty} \hat{x}\, E(\omega_o + \Delta\omega)\, e^{-j\frac{\Delta\omega}{v_g}\left(z - v_g t\right)} \frac{d\,\Delta\omega}{2\pi} \right]$$

Indicates motion at the phase velocity

Indicates motion at the group velocity

- The envelope moves at the group velocity – this is the velocity at which energy in the wave travels
- The oscillating field inside the envelope travels at the phase velocity

Example: A Gaussian Wave Packet



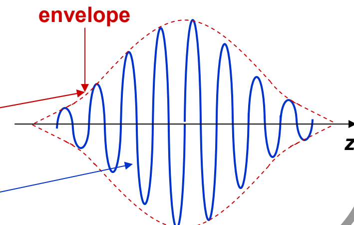
$$E(\omega) = \sqrt{2\pi}\,\tau\, E_o\, e^{-\frac{(\omega-\omega_o)^2\,\tau^2}{2}}$$

$$E(\omega_o + \Delta\omega) = \sqrt{2\pi}\,\tau\, E_o\, e^{-\frac{\Delta\omega^2\,\tau^2}{2}}$$

$$\Rightarrow\quad \vec{E}(z,t) = \mathrm{Re}\left[e^{-j\frac{\omega_o}{v_p}\left(z - v_p t\right)} \int_{-\infty}^{\infty} \hat{x}\, \sqrt{2\pi}\,\tau\, E_o\, e^{-\frac{\Delta\omega^2\,\tau^2}{2}}\, e^{-j\frac{\Delta\omega}{v_g}\left(z - v_g t\right)} \frac{d\,\Delta\omega}{2\pi} \right]$$

$$\Rightarrow\quad \vec{E}(z,t) = \mathrm{Re}\left[e^{-j\frac{\omega_o}{v_p}\left(z - v_p t\right)}\, E_o\, e^{-\frac{\left(z - v_g t\right)^2}{2\left(v_g \tau\right)^2}} \right]$$

$$\Rightarrow\quad \vec{E}(z,t) = E_o\, e^{-\frac{\left(z - v_g t\right)^2}{2\left(v_g \tau\right)^2}} \cos\left[\frac{\omega_o}{v_p}\left(z - v_p t\right) \right]$$



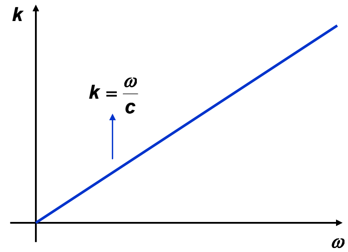
Phase and Group Velocities - III

Examples:

Free Space:

$$\varepsilon(\omega) = \varepsilon_o \qquad k = \frac{\omega}{c}$$

$$v_p = \frac{\omega}{k(\omega)} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c \qquad v_g = \frac{d\omega}{dk} = c$$



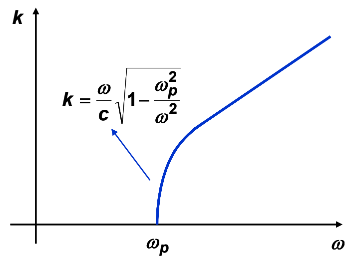
In free space both phase and group velocities are equal to *c*

Plasmas:

$$\varepsilon(\omega) = \varepsilon_o \left[1 - \frac{\omega_p^2}{\omega^2}\right] \qquad k = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

$$v_p = \frac{\omega}{k(\omega)} = \frac{1}{\sqrt{\mu_o \varepsilon(\omega)}} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$



As $\omega \rightarrow \omega_p$ from above, $v_p \rightarrow \infty$ and $v_g \rightarrow 0$

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