Lecture 15

Polarization States of Plane Waves

In this lecture you will learn:

- More complex mathematics for plane waves
- Polarization states of plane waves (linear, circular, elliptical)

Review: Maxwell's Equations for Phasors

Time-harmonic E and H-fields are given as:

$$\vec{E}(\vec{r},t) = \text{Re}\left[\vec{E}(\vec{r})\, e^{j\omega t} \right] \qquad \vec{H}(\vec{r},t) = \text{Re}\left[\vec{H}(\vec{r})\, e^{j\omega t} \right]$$

$$\rho(\vec{r},t) = \text{Re}\left[\rho(\vec{r})\, e^{j\omega t} \right] \qquad \vec{J}(\vec{r},t) = \text{Re}\left[\vec{J}(\vec{r})\, e^{j\omega t} \right]$$

Maxwell's equations for the vector phasors of time-harmonic fields are then:

Gauss' Law:

$$\nabla . \varepsilon_o \vec{E}(\vec{r}) = \rho(\vec{r})$$

Gauss' Law for the Magnetic Field:

$$\nabla . \mu_o \vec{H}(\vec{r}) = 0$$

Faraday's Law:

$$\nabla \times \vec{E}(\vec{r}) = -j\,\omega\,\mu_o\,\vec{H}(\vec{r})$$

Ampere's Law:

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\,\omega\,\varepsilon_o\,\vec{E}(\vec{r})$$

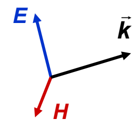
Review: Plane Wave Phasors and Complex Poynting Vector

For a plane wave we know the E-field and H-field phasors to be:

$$\vec{E}(\vec{r}) = \hat{n}\, E_o\, e^{-j\vec{k}.\vec{r}}$$

$$\vec{H}(\vec{r}) = \left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\, e^{-j\vec{k}.\vec{r}}$$

$$\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \approx 377\,\Omega$$



The complex Poynting vector was defined as:

$$\boxed{\vec{S}(\vec{r}) = \vec{E}(\vec{r})\times\vec{H}^*(\vec{r})}$$

The time-average power per unit area is one-half of the real part of the complex Poynting vector

For a plane wave:

$$\left\langle \vec{S}(\vec{r},t)\right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\vec{E}(\vec{r})\times\vec{H}^*(\vec{r})\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\hat{n}\times\left(\hat{k}\times\hat{n}\right)\frac{E_o^2}{\eta_o}\right] = \hat{k}\,\frac{E_o^2}{2\,\eta_o}$$

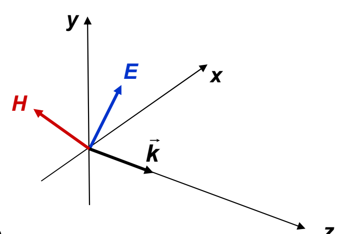
Review: More Calculations in the Complex Notation

Example:

Consider a plane wave with E-field of amplitude E_o and pointing in a direction 45-degrees w.r.t. the x-axis (as shown) and traveling in the +z-direction

Write expression for the E-field phasor:

$$\vec{E}(\vec{r}) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o\, e^{-jkz}$$



Write expression for the H-field phasor:

$$\vec{H}(\vec{r}) = \frac{j}{\omega\,\mu_o}\nabla\times\vec{E}(\vec{r})$$

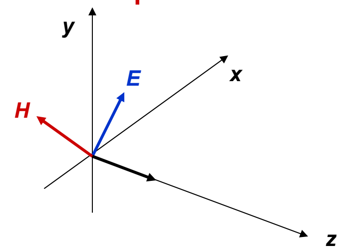
$$\Rightarrow\quad \vec{H}(\vec{r}) = \frac{j}{\omega\,\mu_o}\nabla\times\left(\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)E_o\, e^{-jkz}\right)$$

$$\Rightarrow\quad \vec{H}(\vec{r}) = \frac{j}{\omega\,\mu_o}\left(-jk\,\hat{z}\times\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)\right)E_o\, e^{-jkz} \longrightarrow \left\{\vec{k} = k\,\hat{z}\right.$$

$$\Rightarrow\quad \vec{H}(\vec{r}) = \frac{k}{\omega\,\mu_o}\left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right)E_o\, e^{-jkz}$$

$$\Rightarrow\quad \vec{H}(\vec{r}) = \left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right)\frac{E_o}{\eta_o}\, e^{-jkz}$$

Review: More Calculations in the Complex Notation - II



The E-field phasor is:

$$\vec{E}(\vec{r}) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o \, e^{-jkz}$$

The H-field phasor is:

$$\vec{H}(\vec{r}) = \left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right) \frac{E_o}{\eta_o} \, e^{-jkz}$$

Find the complex Poynting vector:

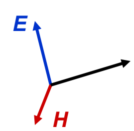
$$\vec{S}(\vec{r}) = \vec{E}(\vec{r}) \times \vec{H}^*(\vec{r}) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) \times \left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right) \frac{E_o^2}{\eta_o}$$

$$= \hat{z}\,\frac{E_o^2}{\eta_o}$$

Find the time-average power per unit area:

$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{2}\mathrm{Re}\left[\vec{S}(\vec{r})\right] = \hat{z}\,\frac{E_o^2}{2\eta_o}$$

Linearly Polarized Plane Waves - I



• **All plane waves of the type:**

$$\vec{E}(\vec{r}) = \hat{n}\, E_o \, e^{-j\vec{k}.\vec{r}} \qquad \vec{H}(\vec{r}) = \left(\hat{k}\times\hat{n}\right)\frac{E_o}{\eta_o}\, e^{-j\vec{k}.\vec{r}}$$

are termed linearly polarized waves

• **The polarized direction is specified by the E-field (by convention) and not by the H-field**

Examples of linearly polarized waves

$$\vec{E}(\vec{r}) = \hat{x}\, E_o \, e^{-jkz} \quad \longleftrightarrow \quad \vec{H}(\vec{r}) = \hat{y}\,\frac{E_o}{\eta_o}\, e^{-jkz}$$

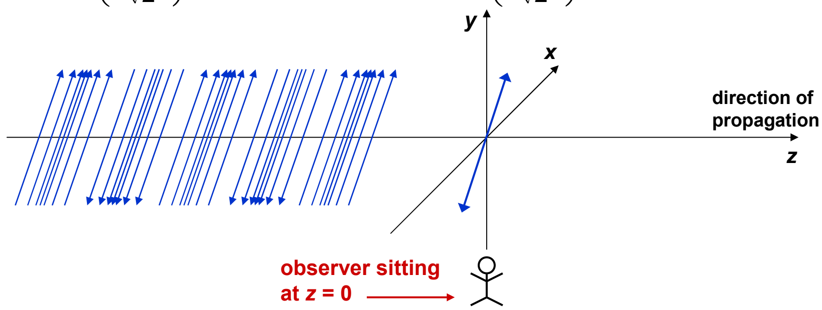
$$\vec{E}(\vec{r}) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o \, e^{-jkz} \quad \longleftrightarrow \quad \vec{H}(\vec{r}) = \left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right)\frac{E_o}{\eta_o}\, e^{-jkz}$$

$$\vec{E}(\vec{r}) = \hat{y}\, E_o \, e^{-j\frac{k}{\sqrt{2}}(x+z)} \quad \longleftrightarrow \quad \vec{H}(\vec{r}) = \left(\frac{-\hat{x}+\hat{z}}{\sqrt{2}}\right)\frac{E_o}{\eta_o}\, e^{-j\frac{k}{\sqrt{2}}(x+z)}$$

Linearly Polarized Plane Waves - II

Consider the linearly polarized plane wave:

$$\vec{E}(\vec{r}) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o \, e^{-j k z} \quad \Rightarrow \quad \vec{E}(\vec{r},t) = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o \cos(\omega t - k z)$$



Observer sitting at $z = 0$ sees an oscillating E-field

$$\vec{E}(\vec{r},t)\Big|_{z=0} = \left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) E_o \cos(\omega t)$$

Circularly Polarized Plane Waves - I

Consider a plane wave given by the phasor:

$$\vec{E}(\vec{r}) = (\hat{x} - j\,\hat{y})\, E_o \, e^{-j k z}$$

$(\hat{x} - j\,\hat{y})$ → A complex polarization vector !!

Lets find the time-dependent E-field:

$$\begin{aligned}
\vec{E}(\vec{r},t) &= \mathrm{Re}\left[\vec{E}(\vec{r})\, e^{j\omega t}\right] \\
&= \mathrm{Re}\left[(\hat{x} - j\,\hat{y})\, E_o \, e^{-j k z} \, e^{j\omega t}\right] \\
&= E_o \, \mathrm{Re}\left[\hat{x}\, e^{-j k z} e^{j\omega t} + \hat{y}\, e^{-j\frac{\pi}{2}} e^{-j k z} e^{j\omega t}\right] \\
&= E_o\left[\hat{x}\cos(\omega t - k z) + \hat{y}\sin(\omega t - k z)\right]
\end{aligned}$$

Notice that the x- and y-components of the E-field have the same amplitude but are 90-degrees out of phase

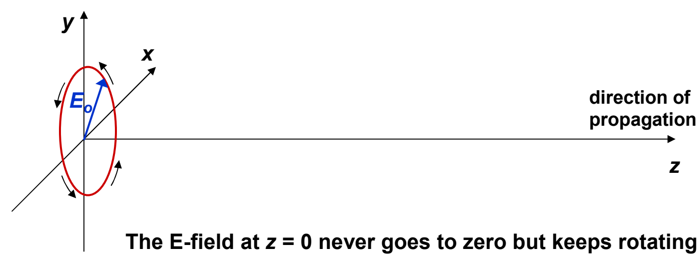
Circularly Polarized Plane Waves - II

$$\vec{E}(\vec{r}) = (\hat{x} - j\,\hat{y})\, E_o\, e^{-jkz} \quad \Rightarrow \quad \vec{E}(\vec{r},t) = E_o\left[\hat{x}\cos(\omega t - kz) + \hat{y}\sin(\omega t - kz)\right]$$

Observer sitting at $z = 0$ will see the E-field vector rotating in a circle

observer sitting at $z = 0$

$$\vec{E}(\vec{r},t)\Big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right]$$



The E-field at $z = 0$ never goes to zero but keeps rotating in a circular trajectory

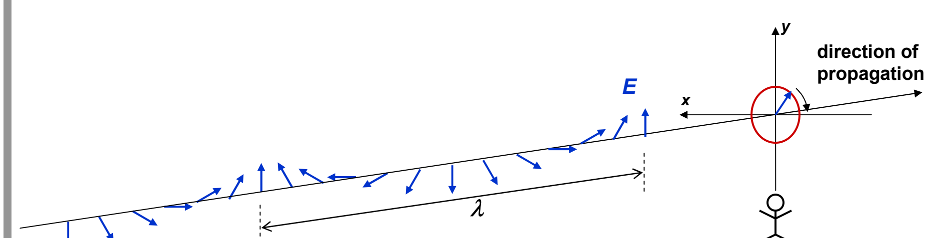
Circularly Polarized Plane Waves - III

So what is happening in Space?

• If one takes a snapshot of a circularly polarized wave at any instant, then he will see the picture shown below

• The E-field vector does not change in magnitude but its direction "twists" in space

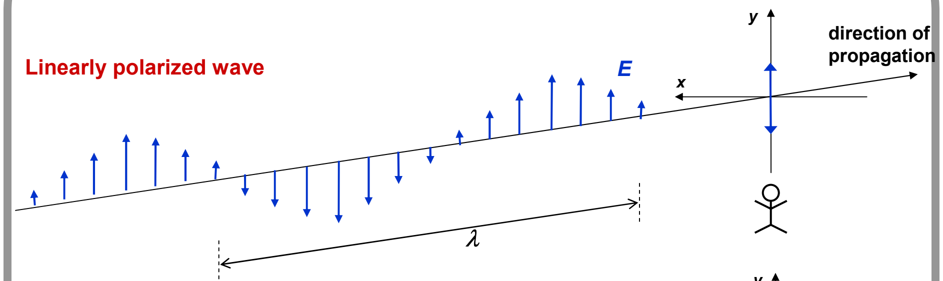
• An observer sitting in the path of the wave will see the E-field vector rotate in a circular trajectory at his location as the wave passes by



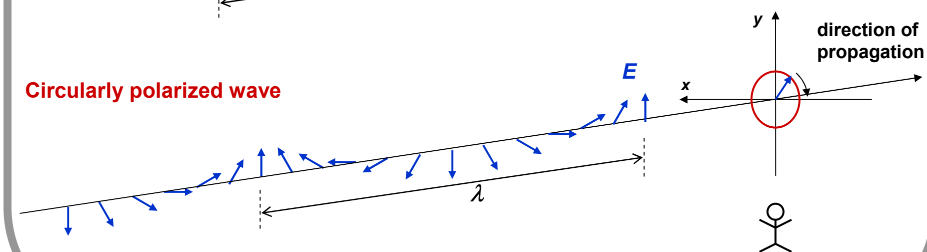
The H-field at each point is orthogonal to the E-field

Linearly Vs Circularly Polarized Plane Waves

Linearly polarized wave



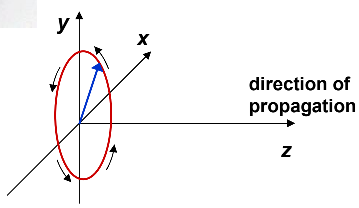
Circularly polarized wave



Handedness of Circularly Polarized Plane Waves

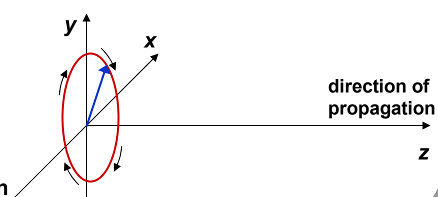
Right-Hand Circular Polarization

The wave is **right-hand circularly polarized** if at any location in space when the thumb of the **right hand** points in the direction of propagation, the fingers curl in the direction of rotation of the E-field vector in a plane perpendicular to the direction of propagation



Left-Hand Circular Polarization

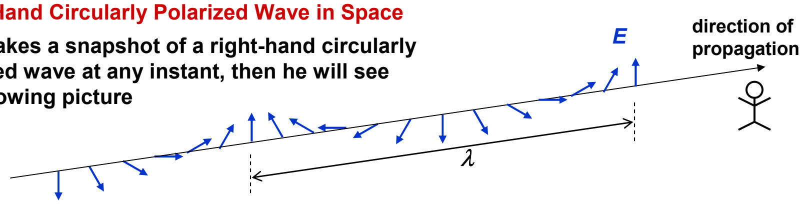
The wave is **left-hand circularly polarized** if at any location in space when the thumb of the **left hand** points in the direction of propagation, the fingers curl in the direction of rotation of the E-field vector in a plane perpendicular to the direction of propagation



Circularly Polarized Plane Waves - IV

Right-Hand Circularly Polarized Wave in Space

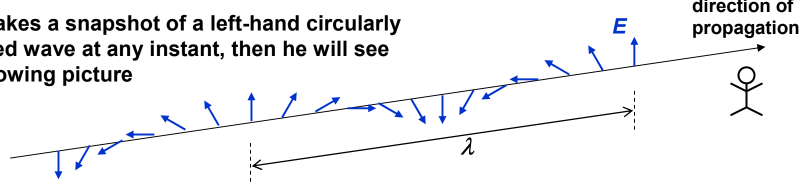
If one takes a snapshot of a right-hand circularly polarized wave at any instant, then he will see the following picture



The H-field at each point is orthogonal to the E-field

Left-Hand Circularly Polarized Wave in Space

If one takes a snapshot of a left-hand circularly polarized wave at any instant, then he will see the following picture



The H-field at each point is orthogonal to the E-field

Elliptically Polarized Plane Waves - I

This is the most general case (all other previously discussed polarization states are specific instances of this general case)

Consider a plane wave moving in the +z-direction and given by the phasor:

$$\vec{E}(\vec{r}) = \left(\hat{x} + A\,e^{j\phi}\,\hat{y}\right) E_o\, e^{-j k z}$$

Notice that the x- and y-components of the E-field have different amplitudes and different phases

$$\begin{aligned}\vec{E}(\vec{r},t) &= \mathrm{Re}\left[\vec{E}(\vec{r})\, e^{j\omega t}\right] \\ &= \mathrm{Re}\left[\left(\hat{x} + A\,e^{j\phi}\,\hat{y}\right) E_o\, e^{-jkz}\, e^{j\omega t}\right] \\ &= E_o\, \mathrm{Re}\left[\hat{x}\, e^{-jkz}\, e^{j\omega t} + \hat{y}\, A\, e^{j\phi}\, e^{-jkz}\, e^{j\omega t}\right] \\ &= E_o\left[\hat{x}\cos(\omega t - kz) + \hat{y}\, A\cos(\omega t - kz + \phi)\right]\end{aligned}$$

At $z = 0$:

$$\vec{E}(\vec{r},t) = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\, A\cos(\omega t + \phi)\right]$$

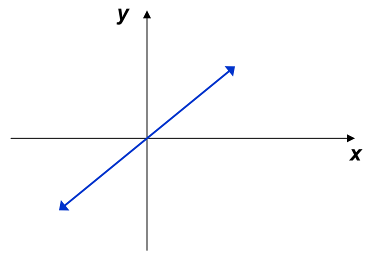
Elliptically Polarized Plane Waves - II

At z = 0: $\vec{E}(\vec{r},t)\big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\,A\cos(\omega t + \phi)\right]$

Example I (A = 1, ϕ = 0):

$$\vec{E}(\vec{r},t)\big|_{z=0} = [\hat{x} + \hat{y}]E_o\cos(\omega t)$$

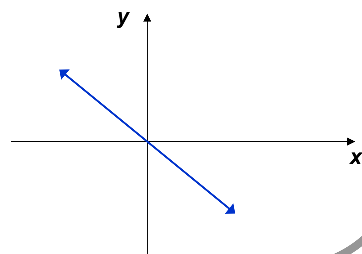
A linearly polarized wave



Example II (A = 1, ϕ = π):

$$\vec{E}(\vec{r},t)\big|_{z=0} = [\hat{x} - \hat{y}]E_o\cos(\omega t)$$

A linearly polarized wave



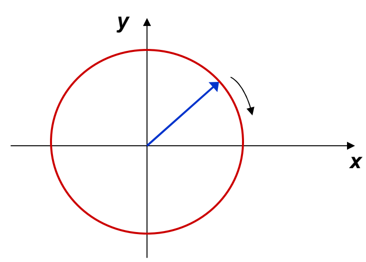
Elliptically Polarized Plane Waves - III

At z = 0: $\vec{E}(\vec{r},t)\big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\,A\cos(\omega t + \phi)\right]$

Example III (A = 1, ϕ = π/2):

$$\vec{E}(\vec{r},t)\big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) - \hat{y}\sin(\omega t)\right]$$

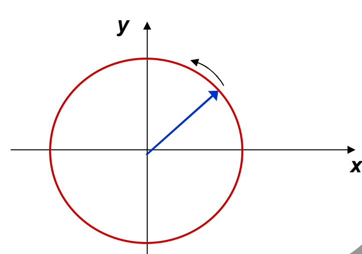
A left-hand circularly polarized wave



Example IV (A = 1, ϕ = -π/2):

$$\vec{E}(\vec{r},t)\big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right]$$

A right-hand circularly polarized wave



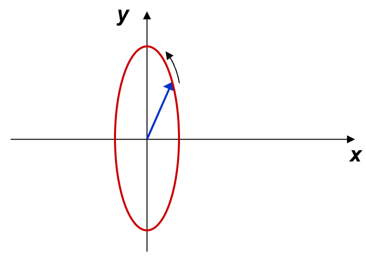
Elliptically Polarized Plane Waves - IV

At z = 0: $\vec{E}(\vec{r},t)\big|_{z=0} = E_o[\hat{x}\cos(\omega t) + \hat{y}\, A\cos(\omega t + \phi)]$

Example V (A = 3, ϕ = $-\pi/2$):

$$\vec{E}(\vec{r},t)\big|_{z=0} = E_o[\hat{x}\cos(\omega t) + \hat{y}\; 3\sin(\omega t)]$$

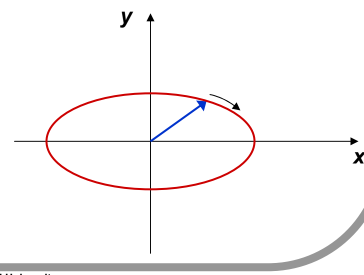
A right-hand elliptically polarized wave



Example VI (A = 0.5, ϕ = $\pi/2$):

$$\vec{E}(\vec{r},t)\big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) - \hat{y}\,\frac{1}{2}\sin(\omega t)\right]$$

A left-hand elliptically polarized wave



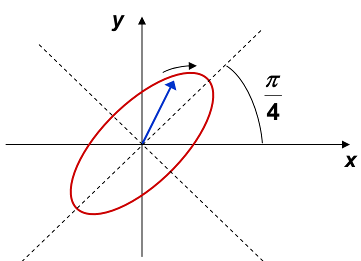
Elliptically Polarized Plane Waves - V

$$\vec{E}(\vec{r},t)\big|_{z=0} = E_o[\hat{x}\cos(\omega t) + \hat{y}\, A\cos(\omega t + \phi)]$$

Example VII (A = 1, ϕ = $\pi/4$):

$$\vec{E}(\vec{r},t)\big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\cos\left(\omega t + \frac{\pi}{4}\right)\right]$$

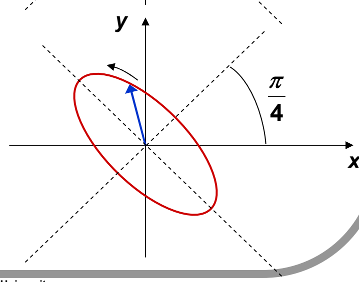
A left-hand elliptically polarized wave



Example VIII (A = 1, ϕ = $-3\pi/4$):

$$\vec{E}(\vec{r},t)\big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\cos\left(\omega t - \frac{3\pi}{4}\right)\right]$$

A right-hand elliptically polarized wave



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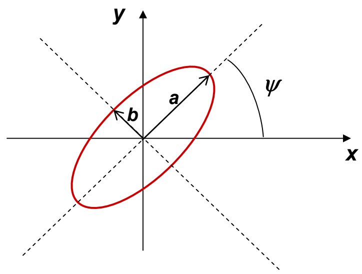
Elliptically Polarized Plane Waves - VI

General Case (A, ϕ):

$$\vec{E}(\vec{r},t)\Big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\, A\cos(\omega t + \phi)\right]$$



Shape and Orientation of the Ellipse:

$$\tan(2\psi) = \frac{2A}{1-A^2}\cos(\phi)$$

$$a^2 + b^2 = 1 + A^2$$

$$ab = A\sin(\phi)$$

If $\chi = \tan^{-1}\left(\frac{b}{a}\right)$ then,

$$\sin(2\chi) = \frac{2A}{1+A^2}\sin(\phi)$$

Handedness:

- If $e^{j\phi}$ is in the upper half of the complex plane then the wave is left-hand elliptically (or circularly) polarized
- If $e^{j\phi}$ is in the lower half of the complex plane then the wave is right-hand elliptically (or circularly) polarized

Linear and/or Circular?

- A circularly polarized wave is a linear superposition of two linearly polarized waves

$$\vec{E}(\vec{r}) = \underbrace{(\hat{x} - j\hat{y})\,E_o\,e^{-jkz}}_{\text{circular}} = \underbrace{\hat{x}\,E_o\,e^{-jkz}}_{\text{linear}} - \underbrace{\hat{y}\;j E_o\,e^{-jkz}}_{\text{linear}}$$

- Similarly, a linearly polarized wave is a linear superposition of two circularly polarized waves

$$\vec{E}(\vec{r}) = \underbrace{\hat{x}\,E_o\,e^{-jkz}}_{\text{linear}} = \underbrace{(\hat{x} + j\hat{y})\frac{E_o}{2}\,e^{-jkz}}_{\text{left-hand circular}} + \underbrace{(\hat{x} - j\hat{y})\frac{E_o}{2}\,e^{-jkz}}_{\text{right-hand circular}}$$

For those familiar with linear algebra

Linear or circular polarization states both form a “complete basis” and any arbitrary polarization state can be expressed as a linear superposition of appropriate basis states chosen from either linear or circular basis