Lecture 15
Polarization States of Plane Waves

In this lecture you will learn:

• More complex mathematics for plane waves
• Polarization states of plane waves (linear, circular, elliptical)

Review: Maxwell’s Equations for Phasors

Time-harmonic E and H-fields are given as:

\[ E(\vec{r}, t) = \text{Re}\left[ E(\vec{r}) e^{j\omega t} \right] \]
\[ H(\vec{r}, t) = \text{Re}\left[ H(\vec{r}) e^{j\omega t} \right] \]
\[ \rho(\vec{r}, t) = \text{Re}\left[ \rho(\vec{r}) e^{j\omega t} \right] \]
\[ J(\vec{r}, t) = \text{Re}\left[ J(\vec{r}) e^{j\omega t} \right] \]

Maxwell’s equations for the vector phasors of time-harmonic fields are then:

Gauss’ Law:

\[ \nabla \cdot \varepsilon_0 \vec{E}(\vec{r}) = \rho(\vec{r}) \]

Gauss’ Law for the Magnetic Field:

\[ \nabla \cdot \mu_0 \vec{H}(\vec{r}) = 0 \]

Faraday’s Law:

\[ \nabla \times \vec{E}(\vec{r}) = -j \omega \mu_0 \vec{H}(\vec{r}) \]

Ampere’s Law:

\[ \nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j \omega \varepsilon_0 \vec{E}(\vec{r}) \]
Review: Plane Wave Phasors and Complex Poynting Vector

For a plane wave we know the E-field and H-field phasors to be:

\[
\begin{align*}
\mathbf{E}(\mathbf{r}) &= \hat{n} E_o e^{-j k \cdot r} \\
\mathbf{H}(\mathbf{r}) &= \left( \mathbf{k} \times \hat{n} \right) \frac{E_o}{\eta_o} e^{-j k \cdot r}
\end{align*}
\]

\[\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 377 \Omega\]

The complex Poynting vector was defined as:

\[
\mathbf{S}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})
\]

The time-average power per unit area is one-half of the real part of the complex Poynting vector

For a plane wave:

\[
\mathbf{\langle S(\mathbf{r}, t) \rangle} = \frac{1}{2} \text{Re}[\mathbf{S}(\mathbf{r})] = \frac{1}{2} \text{Re}[\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})] = \frac{1}{2} \text{Re}\left[ \hat{n} \times (\mathbf{k} \times \hat{n}) \frac{E_o^2}{\eta_o} \right] = k \frac{E_o^2}{2 \eta_o}
\]

Review: More Calculations in the Complex Notation

Example:

Consider a plane wave with E-field of amplitude \(E_o\) and pointing in a direction 45-degrees w.r.t. the x-axis (as shown) and traveling in the +z-direction

Write expression for the E-field phasor:

\[
\mathbf{E}(\mathbf{r}) = \left( \frac{\mathbf{x} + \mathbf{y}}{\sqrt{2}} \right) E_o e^{-j k z}
\]

Write expression for the H-field phasor:

\[
\mathbf{H}(\mathbf{r}) = \frac{j}{\omega \mu_o} \nabla \times \mathbf{E}(\mathbf{r})
\]

\[
\Rightarrow \mathbf{H}(\mathbf{r}) = \frac{j}{\omega \mu_o} \nabla \times \left( \left( \frac{\mathbf{x} + \mathbf{y}}{\sqrt{2}} \right) E_o e^{-j k z} \right)
\]

\[
\Rightarrow \mathbf{H}(\mathbf{r}) = \frac{j}{\omega \mu_o} \left( - j k \mathbf{\hat{z}} \times \left( \frac{\mathbf{x} + \mathbf{y}}{\sqrt{2}} \right) E_o e^{-j k z} \right)
\]

\[
\Rightarrow \mathbf{H}(\mathbf{r}) = \frac{k}{\omega \mu_o} \left( \frac{\mathbf{y} - \mathbf{x}}{\sqrt{2}} \right) E_o e^{-j k z}
\]

\[
\Rightarrow \mathbf{H}(\mathbf{r}) = \left( \frac{\mathbf{y} - \mathbf{x}}{\sqrt{2} \eta_o} \right) E_o e^{-j k z}
\]
Review: More Calculations in the Complex Notation - II

The E-field phasor is:
\[ \mathbf{E}(\mathbf{r}) = \left( \frac{\mathbf{x} + \mathbf{y}}{\sqrt{2}} \right) \mathbf{E}_0 \, e^{-j k z} \]

The H-field phasor is:
\[ \mathbf{H}(\mathbf{r}) = \left( \frac{\mathbf{y} - \mathbf{x}}{\sqrt{2}} \right) \mathbf{H}_0 \, e^{-j k z} / \eta_0 \]

Find the complex Poynting vector:
\[ \mathbf{S}(\mathbf{r}) = \mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r}) = \left( \frac{\mathbf{x} + \mathbf{y}}{\sqrt{2}} \right) \times \left( \frac{\mathbf{y} - \mathbf{x}}{\sqrt{2}} \right) \mathbf{E}_0^2 / \eta_0 \]
\[ = \mathbf{z} \frac{\mathbf{E}_0^2}{\eta_0} \]

Find the time-average power per unit area:
\[ \langle \mathbf{S}(\mathbf{r},t) \rangle = \frac{1}{2} \text{Re} \left[ \mathbf{S}(\mathbf{r}) \right] = \mathbf{z} \frac{\mathbf{E}_0^2}{2 \eta_0} \]

Linearly Polarized Plane Waves - I

• All plane waves of the type:
\[ \mathbf{E}(\mathbf{r}) = \mathbf{n} \mathbf{E}_0 \, e^{-j k.\mathbf{r}} \quad \mathbf{H}(\mathbf{r}) = \left( \mathbf{k} \times \mathbf{n} \right) \mathbf{E}_0 \, e^{-j k.\mathbf{r}} / \eta_0 \]
are termed linearly polarized waves

• The polarized direction is specified by the E-field (by convention) and not by the H-field

Examples of linearly polarized waves
\[ \mathbf{E}(\mathbf{r}) = \mathbf{x} \mathbf{E}_0 \, e^{-j k z} \quad \mathbf{H}(\mathbf{r}) = \mathbf{y} \mathbf{E}_0 \, e^{-j k z} / \eta_0 \]
\[ \mathbf{E}(\mathbf{r}) = \left( \frac{\mathbf{x} + \mathbf{y}}{\sqrt{2}} \right) \mathbf{E}_0 \, e^{-j k z} \quad \mathbf{H}(\mathbf{r}) = \left( \frac{\mathbf{y} - \mathbf{x}}{\sqrt{2}} \right) \mathbf{E}_0 \, e^{-j k z} / \eta_0 \]
\[ \mathbf{E}(\mathbf{r}) = \mathbf{y} \mathbf{E}_0 \, e^{-j k \frac{1}{2} (x+z)} \quad \mathbf{H}(\mathbf{r}) = \left( -\frac{\mathbf{x} + \mathbf{z}}{\sqrt{2}} \right) \mathbf{E}_0 \, e^{-j k \frac{1}{2} (x+z)} / \eta_0 \]
Linearly Polarized Plane Waves - II

Consider the linearly polarized plane wave:

\[ \mathbf{E}(\mathbf{r}) = \left( \frac{\hat{x} + \hat{y}}{\sqrt{2}} \right) E_0 e^{-j k z} \Rightarrow \mathbf{E}(\mathbf{r}, t) = \left( \frac{\hat{x} + \hat{y}}{\sqrt{2}} \right) E_0 \cos(\omega t - k z) \]

Observer sitting at \( z = 0 \) sees an oscillating E-field

\[ \mathbf{E}(\mathbf{r}, t)_{z=0} = \left( \frac{\hat{x} + \hat{y}}{\sqrt{2}} \right) E_0 \cos(\omega t) \]

Circularly Polarized Plane Waves - I

Consider a plane wave given by the phasor:

\[ \mathbf{E}(\mathbf{r}) = (\hat{x} - j \hat{y}) E_0 e^{-j k z} \]

A complex polarization vector !!

Let's find the time-dependent E-field:

\[
\mathbf{E}(\mathbf{r}, t) = \text{Re} \left[ \mathbf{E}(\mathbf{r}) e^{j \omega t} \right] \\
= \text{Re} \left[ (\hat{x} - j \hat{y}) E_0 e^{-j k z} e^{j \omega t} \right] \\
= E_0 \text{Re} \left[ \hat{x} e^{-j k z} e^{j \omega t} - j \hat{y} e^{-j k z} e^{j \omega t} \right] \\
= E_0 \left[ \hat{x} \cos(\omega t - k z) + \hat{y} \sin(\omega t - k z) \right]
\]

Notice that the x- and y-components of the E-field have the same amplitude but are 90-degrees out of phase.
Circularly Polarized Plane Waves - II

\[
\vec{E}(\vec{r}) = (\hat{x} - j \hat{y}) E_o e^{-j k z} \Rightarrow \vec{E}(\vec{r}, t) = E_o [\hat{x} \cos(\omega t - k z) + \hat{y} \sin(\omega t - k z)]
\]

Observer sitting at z = 0 will see the E-field vector rotating in a circle

Observer sitting at z = 0 will see the E-field vector rotating in a circle

The E-field at z = 0 never goes to zero but keeps rotating in a circular trajectory

Circularly Polarized Plane Waves - III

So what is happening in Space?

- If one takes a snapshot of a circularly polarized wave at any instant, then he will see the picture shown below

- The E-field vector does not change in magnitude but its direction “twists” in space

- An observer sitting in the path of the wave will see the E-field vector rotate in a circular trajectory at his location as the wave passes by

The H-field at each point is orthogonal to the E-field
**Linearly Vs Circularly Polarized Plane Waves**

**Linearly polarized wave**

**Circularly polarized wave**

**Handedness of Circularly Polarized Plane Waves**

**Right-Hand Circular Polarization**

The wave is right-hand circularly polarized if at any location in space when the thumb of the right hand points in the direction of propagation, the fingers curl in the direction of rotation of the E-field vector in a plane perpendicular to the direction of propagation.

**Left-Hand Circular Polarization**

The wave is left-hand circularly polarized if at any location in space when the thumb of the left hand points in the direction of propagation, the fingers curl in the direction of rotation of the E-field vector in a plane perpendicular to the direction of propagation.
Circularly Polarized Plane Waves - IV

Right-Hand Circularly Polarized Wave in Space
If one takes a snapshot of a right-hand circularly polarized wave at any instant, then he will see the following picture.

The H-field at each point is orthogonal to the E-field.

Left-Hand Circularly Polarized Wave in Space
If one takes a snapshot of a left-hand circularly polarized wave at any instant, then he will see the following picture.

The H-field at each point is orthogonal to the E-field.

Elliptically Polarized Plane Waves - I

This is the most general case (all other previously discussed polarization states are specific instances of this general case).

Consider a plane wave moving in the +z-direction and given by the phasor:

\[
\mathbf{E}(r) = \left( \hat{x} + A e^{j \theta} \hat{y} \right) E_0 e^{-jkz}
\]

Notice that the x- and y-components of the E-field have different amplitudes and different phases.

\[
\mathbf{E}(r, t) = \text{Re} \left[ \mathbf{E}(r) e^{j \omega t} \right] = \text{Re} \left[ \hat{x} + A e^{j \theta} \hat{y} \right] E_0 e^{-jkz} e^{j \omega t}
\]

At \( z = 0 \):

\[
\mathbf{E}(r, t) = E_0 [\hat{x} \cos(\omega t - kz) + \hat{y} \cos(\omega t - kz + \phi)]
\]
Elliptically Polarized Plane Waves - II

At $z = 0$: 
$$ E(\bar{r}, t)_{z=0} = E_0 [\hat{x} \cos (\omega t) + \hat{y} \cos (\omega t + \phi)] $$

Example I ($A = 1, \phi = 0$):
$$ E(\bar{r}, t)_{z=0} = [\hat{x} + \hat{y}]E_0 \cos (\omega t) $$
A linearly polarized wave

Example II ($A = 1, \phi = \pi$):
$$ E(\bar{r}, t)_{z=0} = [\hat{x} - \hat{y}]E_0 \cos (\omega t) $$
A linearly polarized wave

Elliptically Polarized Plane Waves - III

At $z = 0$: 
$$ E(\bar{r}, t)_{z=0} = E_0 [\hat{x} \cos (\omega t) + \hat{y} \cos (\omega t + \phi)] $$

Example III ($A = 1, \phi = \pi/2$):
$$ E(\bar{r}, t)_{z=0} = E_0 [\hat{x} \cos (\omega t) - \hat{y} \sin (\omega t)] $$
A left-hand circularly polarized wave

Example IV ($A = 1, \phi = \pi/2$):
$$ E(\bar{r}, t)_{z=0} = E_0 [\hat{x} \cos (\omega t) + \hat{y} \sin (\omega t)] $$
A right-hand circularly polarized wave
Elliptically Polarized Plane Waves - IV

At \( z = 0 \): \( \vec{E}(r, t)\big|_{z=0} = E_0 \left[ \hat{x} \cos(\omega t) + \hat{y} A \cos(\omega t + \phi) \right] \)

Example V (\( A = 3, \phi = -\pi/2 \)):
\[
\vec{E}(r, t)\big|_{z=0} = E_0 \left[ \hat{x} \cos(\omega t) + \hat{y} 3 \sin(\omega t) \right]
\]
A right-hand elliptically polarized wave

Example VI (\( A = 0.5, \phi = \pi/2 \)):
\[
\vec{E}(r, t)\big|_{z=0} = E_0 \left[ \hat{x} \cos(\omega t) - \hat{y} \frac{1}{2} \sin(\omega t) \right]
\]
A left-hand elliptically polarized wave

Elliptically Polarized Plane Waves - V

\[
\vec{E}(r, t)\big|_{z=0} = E_0 \left[ \hat{x} \cos(\omega t) + \hat{y} A \cos(\omega t + \phi) \right]
\]

Example VII (\( A = 1, \phi = \pi/4 \)):
\[
\vec{E}(r, t)\big|_{z=0} = E_0 \left[ \hat{x} \cos(\omega t) + \hat{y} \cos\left(\omega t + \frac{\pi}{4}\right) \right]
\]
A left-hand elliptically polarized wave

Example VIII (\( A = 1, \phi = -3\pi/4 \)):
\[
\vec{E}(r, t)\big|_{z=0} = E_0 \left[ \hat{x} \cos(\omega t) + \hat{y} \cos\left(\omega t - \frac{3\pi}{4}\right) \right]
\]
A right-hand elliptically polarized wave
Elliptically Polarized Plane Waves - VI

General Case ($A$, $\phi$):

$$E(r, t)_{z=0} = E_0 [\hat{x} \cos(\omega t) + \hat{y} A \cos(\omega t + \phi)]$$

Shape and Orientation of the Ellipse:

$$\tan(2\psi) = \frac{2A}{1-A^2} \cos(\phi)$$

$$a^2 + b^2 = 1 + A^2$$

$$ab = A \sin(\phi)$$

If $\chi = \tan^{-1}\left(\frac{b}{a}\right)$ then,

$$\sin(2\chi) = \frac{2A}{1+4A^2} \sin(\phi)$$

Handedness:

- If $e^{i\phi}$ is in the upper half of the complex plane then the wave is left-hand elliptically (or circularly) polarized
- If $e^{i\phi}$ is in the lower half of the complex plane then the wave is right-hand elliptically (or circularly) polarized

Linear and/or Circular?

- A circularly polarized wave is a linear superposition of two linearly polarized waves

$$\vec{E}(r) = (\hat{x} - j\hat{y}) E_0 e^{-jkz} = \hat{x} E_0 e^{-jkz} - \hat{y} j E_0 e^{-jkz}$$

- Similarly, a linearly polarized wave is a linear superposition of two circularly polarized waves

$$\vec{E}(r) = \hat{x} E_0 e^{-jkz} = (\hat{x} + j\hat{y}) \frac{E_0}{2} e^{-jkz} + (\hat{x} - j\hat{y}) \frac{E_0}{2} e^{-jkz}$$

For those familiar with linear algebra

Linear or circular polarization states both form a “complete basis” and any arbitrary polarization state can be expressed as a linear superposition of appropriate basis states chosen from either linear or circular basis.