Lecture 12

Energy, Force, and Work in Electro- and Magneto-Quasistatics

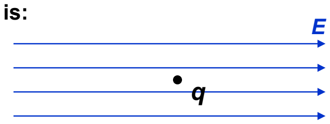
In this lecture you will learn:

- Relationship between energy, force, and work in electroquasistatic and magnetoquasistatic systems

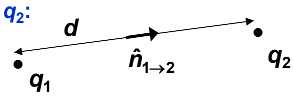
Lorentz Electric Force

The force on a charge q in an E-field is:

$$\vec{F} = q\vec{E}$$



Example: consider two charges q_1 and q_2:



The force that q_1 exerts upon q_2 can be obtained by multiplying the E-field that q_1 produces at the location of q_2 by the charge q_2 :

$$\vec{F}_{1\rightarrow 2} = q_2\,\vec{E}_1$$

$$= q_2 \frac{q_1}{4\pi\,\varepsilon_o\,d^2}\,\hat{n}_{1\rightarrow 2} = \frac{q_1\,q_2}{4\pi\,\varepsilon_o\,d^2}\,\hat{n}_{1\rightarrow 2}$$

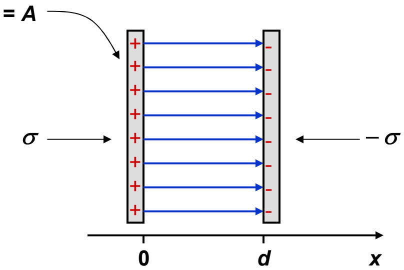
The force that q_2 exerts upon q_1 is just equal and opposite to what q_1 exerts upon q_2 (Newton's third law):

$$\vec{F}_{2\rightarrow 1} = -\frac{q_1\,q_2}{4\pi\,\varepsilon_o\,d^2}\,\hat{n}_{1\rightarrow 2} = \frac{q_1\,q_2}{4\pi\,\varepsilon_o\,d^2}\,\hat{n}_{2\rightarrow 1}$$

Force Between Charged Plates – I

Consider two charged plates carrying charge per unit areas of $+\sigma$ and $-\sigma$, respectively

Need to find the total force on the right plate exerted by the left plate (which is also equal an opposite to the force on the left plate exerted by the right plate)



Step (1): Find the E-field at the location of the right plate produced by the left plate ASSUMING THE RIGHT PLATE WAS NOT THERE:

$$E_x(x=d)\big|_{no\ right\ plate} = \frac{\sigma}{2\,\varepsilon_o}$$

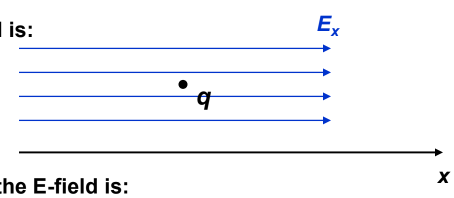
Step (2): Multiply the E-field calculated above by the total charge on the right plate to get the desired force

$$F_x\big|_{L\to R} = (-A\sigma)\,E_x(x=d)\big|_{no\ right\ plate} = -\frac{A\sigma^2}{2\,\varepsilon_o}$$

Potential Energy and Force

The force on a charge q in an E-field is:

$$F_x = qE_x$$



The potential energy of a charge in the E-field is:

$$U = q\,\phi(x) = -qE_x\,x$$

So the force on the charge can also be written as:

$$F_x = -\frac{dU}{dx}$$

Electric force is the derivative of potential energy

This formula is much more general than it appears (see the next few slides …..)

Energy, Force, and Work in Electromagnetics

Lets generalize the relationship between energy and force

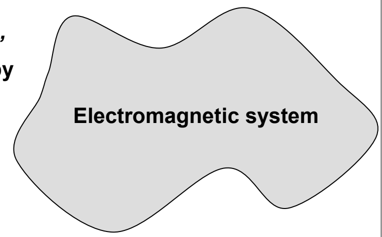
Suppose one has a CLOSED electromagnetic system with electromagnetic energy U

If the system does some mechanical work "$F\,dx$" then its electromagnetic energy must decrease by the same amount:

$$dU = -F dx$$

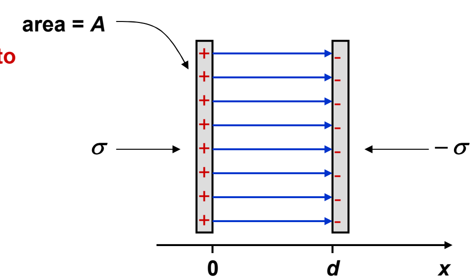
or

$$F = -\frac{dU}{dx}$$



Here "F" is some force that results in the change of some system length parameter by "dx"

Force Between Charged Plates - II



Now lets use the energy concepts to calculate the same force

Total electric energy stored in the field can be calculated as follows:

$$U = \iiint \frac{\varepsilon_o \vec{E}.\vec{E}}{2} dV$$

$$= \frac{\varepsilon_o}{2}\left(\frac{\sigma}{\varepsilon_o}\right)^2 (A\,d)$$

The force between the plates can move the plates thereby doing work that would result in a change of the value of "d"

$$F = -\frac{\partial U}{\partial d}$$

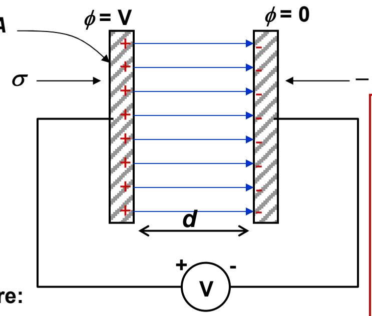
$$= -\frac{\varepsilon_o}{2}\left(\frac{\sigma}{\varepsilon_o}\right)^2 A = -\frac{A\sigma^2}{2\,\varepsilon_o}$$

The negative sign indicates the force is in the direction of decreasing "d" i.e. the force between the plates is attractive

Force Between Charged Metal Plates - I

Now consider the force between the charged metal plates of a parallel plate capacitor connected to a voltage source

What is the force between the plates in terms of the applied voltage *V*?



$$E_x(x) = \frac{V}{d}$$

$$Q = CV$$

$$C = \frac{\varepsilon_o A}{d}$$

$$U = \frac{1}{2}CV^2$$

$$\sigma = \varepsilon_o \frac{V}{d}$$

Answer should be the same as before:

$$F = -\frac{A\sigma^2}{2\varepsilon_o} = -\frac{1}{2}\frac{CV^2}{d}$$

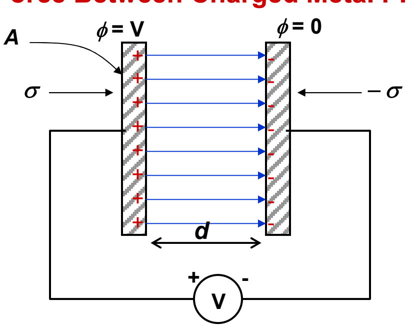
But now when you use the previous formula you will get the wrong sign:

$$F = -\frac{\partial U}{\partial d} = -\frac{1}{2}\frac{\partial C}{\partial d}V^2 = \frac{1}{2}\frac{CV^2}{d} \quad \times$$

What went wrong ?

The presence of a voltage source means you don't have a closed system anymore

Force Between Charged Metal Plates - II



$$E_x(x) = \frac{V}{d}$$

$$Q = CV$$

$$C = \frac{\varepsilon_o A}{d}$$

$$U = \frac{1}{2}CV^2$$

$$\sigma = \varepsilon_o \frac{V}{d}$$

- If the system tries to do work by bringing the plates closer or pushing them further apart, the voltage source ensures that the charge *Q* in the capacitor always satisfies the relation: *Q=CV*
- The voltage source ensures that *Q* is always equal to *CV* by bringing in or removing charge from the capacitor while the mechanical work is being performed
- This work done by the voltage source in bringing in or removing charges must also be included in the analysis

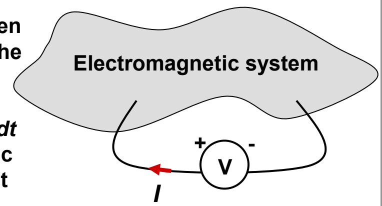
Energy, Force, and Work in Electromagnetics – Voltage Sources

Lets further generalize the relationship between energy and force by including voltage sources

Suppose one has an electromagnetic system with electromagnetic energy *U* that is connected to a FIXED voltage source *V*

- **If the system performs mechanical work *Fdx* then its electromagnetic energy *U* must decrease by the same amount**

- **If the voltage source passes a current *I* in time *dt* then it does work and the system electromagnetic energy *U* must also increase by the same amount**



So instead of: $dU = -F dx$

We write: $dU = -F dx + I V\, dt$

But if *dQ* is the total charge that passed in time *dt* then: $dQ = I\, dt$

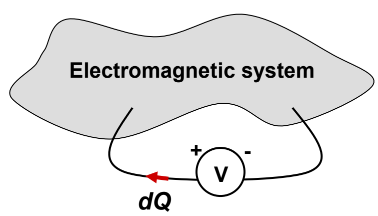
$$\Rightarrow \quad dU = -F dx + V\, dQ$$

$$\Rightarrow \quad F = -\frac{\partial U}{\partial x} + V \frac{\partial Q}{\partial x}$$ **Assuming voltage *V* is held fixed**

Energy, Force, and Work in Electromagnetics – Voltage Sources

More generally the total differential of energy *U* can be written as:

$$dU = \left.\frac{\partial U}{\partial x}\right|_{Q \text{ fixed}} dx + \left.\frac{\partial U}{\partial Q}\right|_{x \text{ fixed}} dQ$$



But we had:

$$dU = -F dx + V dQ$$

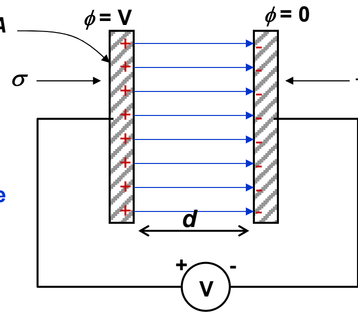
Therefore, it must be that:

$$F = -\left.\frac{\partial U}{\partial x}\right|_{Q \text{ fixed}} \quad \text{and} \quad V = \left.\frac{\partial U}{\partial Q}\right|_{x \text{ fixed}}$$

But if the thing that is kept fixed is the voltage *V* then we already have the result:

$$F = \left.\left(-\frac{\partial U}{\partial x} + V \frac{\partial Q}{\partial x}\right)\right|_{V \text{ fixed}}$$

Force Between Charged Metal Plates - III



Electric energy is:

$$U = \frac{1}{2} C V^2$$

Since the voltage *V* is held fixed the force between the plates can be calculated as:

$$F = \left(-\frac{\partial U}{\partial d} + V \frac{\partial Q}{\partial d} \right) \Bigg|_{V \text{ fixed}}$$

$$= -\frac{1}{2} \frac{\partial C}{\partial d} V^2 + V \frac{\partial C}{\partial d} V$$

$$= -\frac{1}{2} \frac{C V^2}{d}$$

This time we have got the sign right as well

$$E_x(x) = \frac{V}{d}$$

$$Q = CV$$

$$C = \frac{\varepsilon_o A}{d}$$

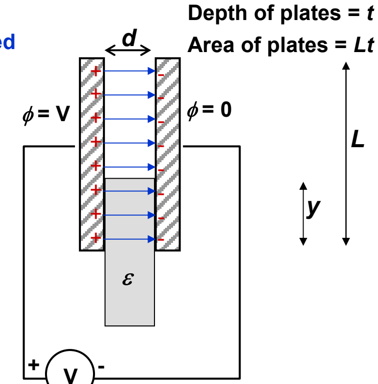
$$U = \frac{1}{2} C V^2$$

$$\sigma = \varepsilon_o \frac{V}{d}$$

Application: An Electrostatic Actuator

The applied voltage can be used to pull in the dielectric slab and when the voltage is removed the slab will come down by gravity

Depth of plates = *t*
Area of plates = *Lt*



Electric energy is:

$$U = \frac{1}{2} C V^2$$

Force on the slab is:

$$F = \left(-\frac{\partial U}{\partial y} + V \frac{\partial Q}{\partial y} \right) \Bigg|_{V \text{ fixed}}$$

$$= -\frac{1}{2} \frac{\partial C}{\partial y} V^2 + V \frac{\partial C}{\partial y} V$$

$$= \frac{1}{2} \frac{(\varepsilon - \varepsilon_o) t}{d} V^2$$

+ve sign of the force means that the force is in the direction of increasing *y*, i.e. the slab will be pulled in when a voltage is applied

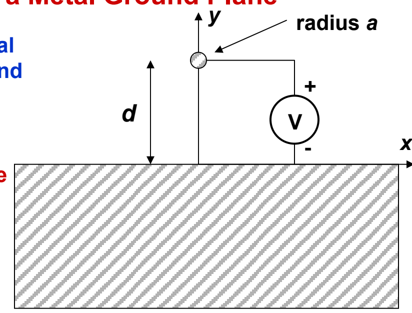
$$E_x = \frac{V}{d} \qquad Q = CV$$

$$C = \frac{\varepsilon_o (L - y) t + \varepsilon y t}{d}$$

$$U = \frac{1}{2} C V^2$$

Force on a Metal Wire Over a Metal Ground Plane

Consider a very long (in the z-direction) metal wire of radius *a* over a metal ground plane and at a distance d/2 from it



Need to find the force per unit length between the metal wire and the ground plane when the potential difference between them is *V* volts

Electric energy per unit length is:

$$U = \frac{1}{2} C V^2$$

Force per unit length on the wire is:

$$F = \left(-\frac{\partial U}{\partial d} + V \frac{\partial Q}{\partial d} \right)\Bigg|_{V \text{ fixed}}$$

$$= -\frac{1}{2}\frac{\partial C}{\partial d} V^2 + V \frac{\partial C}{\partial d} V$$

$$= \frac{1}{2}\frac{\partial C}{\partial d} V^2 = -\frac{1}{2} \frac{C V^2}{d \ln\left(\frac{2d}{a}\right)}$$

From homework #2

$$Q = CV$$

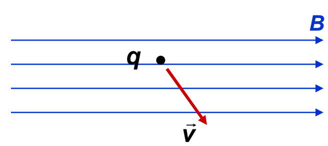
$$C = \frac{2\pi\varepsilon_o}{\ln\left(\frac{2d}{a}\right)}$$

$$U = \frac{1}{2} C V^2$$

Lorentz Magnetic Force

The force on a charge *q* moving in a magnetic field is:

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

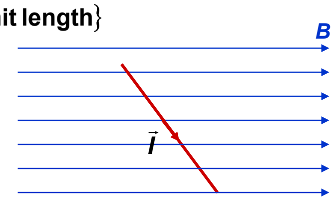


The force per unit length on a wire carrying current *I* is:

$$\vec{F} = \left\{ q\,\vec{v} \times \vec{B} \right\} . \left\{ \text{number of charges per unit length} \right\}$$

But: $\vec{I} = \left\{ q\,\vec{v} \right\} . \left\{ \text{number of charges per unit length} \right\}$

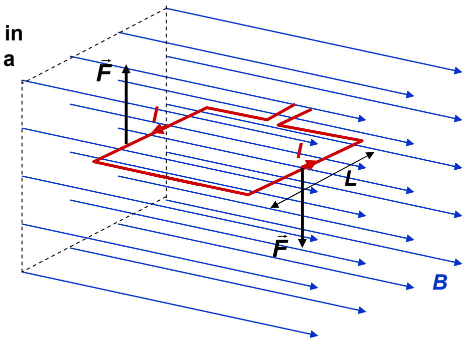
$$\Rightarrow \quad \vec{F} = \vec{I} \times \vec{B}$$



Electric Motors

A coil of a current carrying wire in a magnetic field will experience a torque as its two ends will experience forces in opposite directions

$$\vec{F} = \left(\vec{I} \times \vec{B}\right) L$$



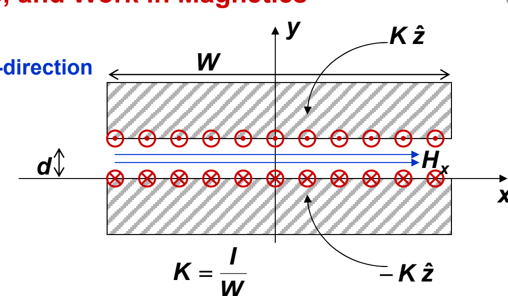
Energy, Forces, and Work in Magnetics

The structure has length ℓ in the z-direction
Recall that:

$$H_x = K = \frac{I}{W}$$

$$\Rightarrow L = \frac{\mu_o H_x \, d\,\ell}{I} = \frac{\mu_o \, d\ell}{W}$$

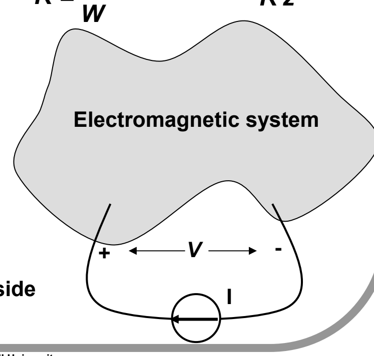
(*L* is the total inductance)



Question: What is the force between the two metal plates? Is it repulsive or is it attractive?

Try the same procedure: if a system does some mechanical work *Fdx* then its electromagnetic energy must decrease by the same amount:

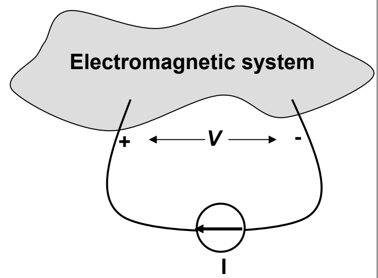
$$dU = -Fdx \quad \text{or} \quad F = -\frac{dU}{dx}$$



Magnetic systems that require current from outside sources are NOT REALLY CLOSED SYSTEMS

Energy, Forces, and Work in Electromagnetics – Current Sources

Suppose one has an electromagnetic system with electromagnetic energy *U* that is connected to a FIXED current source *I*



- If the system performs mechanical work *Fdx* then its electromagnetic energy *U* must decrease by the same amount
- If the current source passes current *I* under a potential *V* in time *dt* then it does work and the system electromagnetic energy *U* must also increase by the same amount

So instead of: $dU = -F dx$

We write: $dU = -F dx + I V dt$

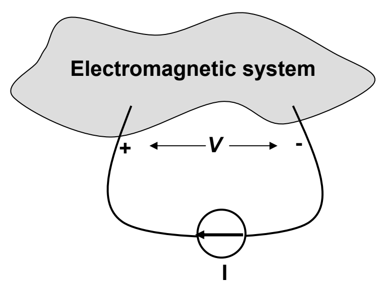
But if the electromagnetic system is an inductor then we know that: $V = \frac{d\lambda}{dt}$

$\Rightarrow \quad dU = -F dx + I d\lambda$ → The product $I d\lambda$ is the work done by the current source in changing the flux by $d\lambda$

$\Rightarrow \quad F = -\frac{\partial U}{\partial x} + I \frac{\partial \lambda}{\partial x}$ → **Assuming current *I* is held fixed**

Energy, Forces, and Work in Electromagnetics – Current Sources

More generally the total differential of energy *U* can be written as:



$$dU = \left.\frac{\partial U}{\partial x}\right|_{\lambda \text{ fixed}} dx + \left.\frac{\partial U}{\partial \lambda}\right|_{x \text{ fixed}} d\lambda$$

But we had:

$$dU = -F dx + I d\lambda$$

Therefore, it must be that:

$$F = -\left.\frac{\partial U}{\partial x}\right|_{\lambda \text{ fixed}} \quad \text{and} \quad I = \left.\frac{\partial U}{\partial \lambda}\right|_{x \text{ fixed}}$$

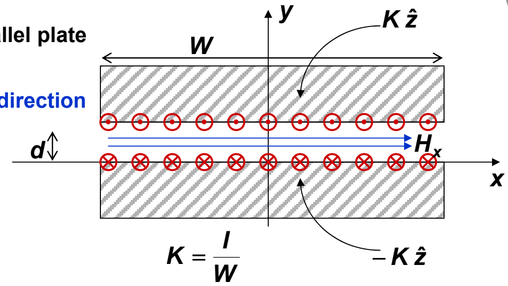
But if the thing that is kept fixed is the current *I* then we already have the result:

$$F = \left.\left(-\frac{\partial U}{\partial x} + I \frac{\partial \lambda}{\partial x} \right)\right|_{I \text{ fixed}}$$

Force Between Parallel Current Carrying Plates

Force between the plates in a parallel plate inductor

The structure has length ℓ in the z-direction



$$H_x = K = \frac{I}{W}$$

$$\lambda = LI$$

$$L = \frac{\mu_o d\ell}{W}$$

$$U = \frac{1}{2}LI^2$$

$$F = \left(-\frac{\partial U}{\partial d} + I\frac{\partial \lambda}{\partial d}\right)\Bigg|_{I \text{ fixed}}$$

$$\Rightarrow F = \frac{1}{2}\frac{\partial L}{\partial d}I^2 = \frac{1}{2}\frac{L}{d}I^2$$

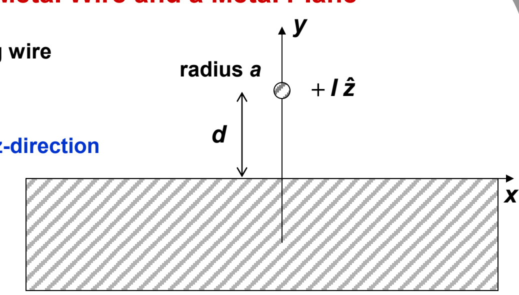
The +ve sign indicates that the force is in the direction of increasing "*d*"

$\Rightarrow$ Force is repulsive - the plates repel each other

Force Between a Metal Wire and a Metal Plane

Force between a current carrying wire and a perfect metal plane

The structure has length ℓ in the z-direction



$$\lambda = LI$$

$$L = \frac{\mu_o \ell}{2\pi}\log\left(\frac{2d}{a}\right)$$

$$U = \frac{1}{2}LI^2$$

$$F = \left(-\frac{\partial U}{\partial d} + I\frac{\partial \lambda}{\partial d}\right)\Bigg|_{I \text{ fixed}}$$

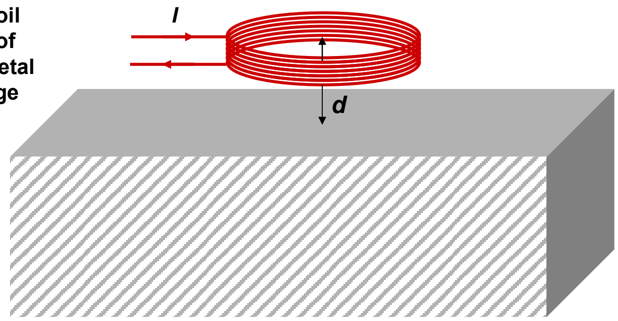
$$\Rightarrow F = \frac{1}{2}\frac{\partial L}{\partial d}I^2 = \frac{1}{2}\frac{L}{d\log\left(\frac{2d}{a}\right)}I^2$$

The +ve sign indicates that the force is in the direction of increasing "*d*"

$\Rightarrow$ Force is repulsive – the metal wire is repelled by the metal plane

Application: Magnetic Levitation

The inductance *L* of the coil is an increasing function of the distance *d* from the metal block (…. because of image currents)



$$\lambda = LI$$

$$L = L(d)$$

$$U = \frac{1}{2}LI^2$$

$$F = \left(-\frac{\partial U}{\partial d} + I\frac{\partial \lambda}{\partial d}\right)\Bigg|_{I \text{ fixed}}$$

$$\Rightarrow F = \frac{1}{2}\frac{\partial L}{\partial d}I^2$$

The +ve sign indicates that the force is in the direction of increasing "*d*"

⇒ Force is repulsive – the metal coil is repelled by the metal plane