Lecture 10

Perfect Metals in Magnetism and Inductance

In this lecture you will learn:

- Some more about the vector potential
- Magnetic field boundary conditions
- The behavior of perfect metals towards time-varying magnetic fields
- Image currents and magnetic diffusion
- Inductance

The Vector Potential - Review

In electroquasistatics we had: $\nabla \times \vec{E} = 0$

Therefore we could represent the E-field by the scalar potential: $\vec{E} = -\nabla \phi$

In magnetoquasistatics we have: $\nabla . (\vec{B}) = \nabla . (\mu_o \vec{H}) = 0$

Therefore we can represent the B-field by the vector potential:

$$\vec{B} = \mu_o \vec{H} = \nabla \times \vec{A}$$

A vector field can be specified (up to a constant) by specifying its curl and its divergence

Our definition of the vector potential $\vec{A}$ is not yet unique – we have only specified its curl

For simplicity we fix the divergence of the vector potential $\vec{A}$ to be zero:

$$\nabla . \vec{A} = 0$$

Magnetic Flux and Vector Potential Line Integral - Review

The magnetic flux λ through a surface is the surface integral of the B-field through the surface

$$\lambda = \iint \vec{B}\,.\,d\vec{a}$$
$$= \mu_o \iint \vec{H}\,.\,d\vec{a}$$

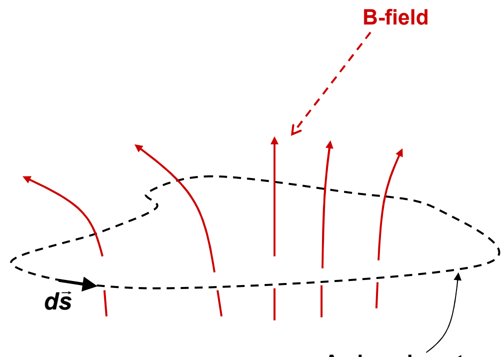
Since:

$$\vec{B} = \mu_o \vec{H} = \nabla \times \vec{A}$$

We get:

$$\lambda = \iint \vec{B}\,.\,d\vec{a}$$
$$= \iint \left(\nabla \times \vec{A}\right).\,d\vec{a}$$
$$= \oint \vec{A}\,.\,d\vec{s}$$

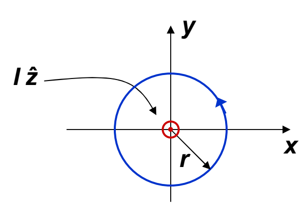
Stoke's Theorem



The magnetic flux through a surface is equal to the line-integral of the vector potential along a closed contour bounding that surface

Vector Potential of a Line-Current

Consider an infinitely long line-current with current I in the +z-direction



Work in cylindrical co-ordinates

The H-field has only a ϕ-component

Using Ampere's Law: $H_\phi\left(2\pi\, r\right) = I$

$$\Rightarrow \quad H_\phi = \frac{I}{2\pi\, r}$$

$$\nabla^2 \vec{A} = -\mu_o \vec{J}$$

If the current has only a z-component then the vector potential also only has a z-component which, by symmetry, is only a function of the distance from the line-current

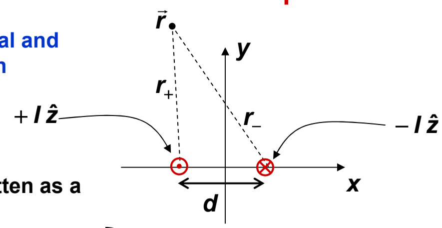
But $$H_\phi = \frac{\nabla \times \vec{A}}{\mu_o} = -\frac{1}{\mu_o}\frac{\partial A_z(r)}{\partial r} \quad \Rightarrow \quad \frac{\partial A_z(r)}{\partial r} = -\frac{\mu_o I}{2\pi\, r}$$

Integrating from r_o to r :

$$A_z(r) - A_z(r_o) = \frac{\mu_o I}{2\pi}\ln\left(\frac{r_o}{r}\right)$$

Vector Potential of a Line-Current Dipole

Consider two infinitely long equal and opposite line-currents, as shown



The vector potential can be written as a sum using superposition:

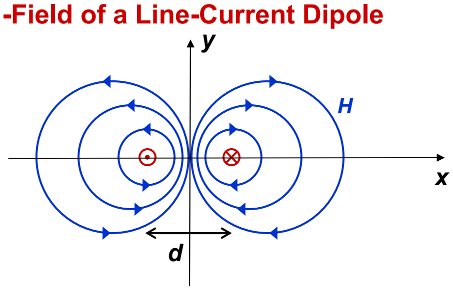
$$A_z(\vec{r}) = \frac{\mu_o I}{2\pi}\ln\left(\frac{r_o}{r_+}\right) - \frac{\mu_o I}{2\pi}\ln\left(\frac{r_o}{r_-}\right) = \frac{\mu_o I}{2\pi}\ln\left(\frac{r_-}{r_+}\right)$$

The final answer does not depend on the parameter r_o

Question: where is the zero of the vector potential?

Points for which r_+ equals r_- have zero potential. These points constitute the entire *y-z* plane

H-Field of a Line-Current Dipole



Something to Ponder Upon

Poisson equation:

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_o}$$

Potential of a line-charge dipole:

$$\phi(\vec{r}) = \frac{\lambda}{2\pi\,\varepsilon_o}\ln\left(\frac{r_-}{r_+}\right)$$

Vector Poisson equation (only the z-component relevant for this problem)

$$\nabla^2 A_z = -\mu_o J_z$$

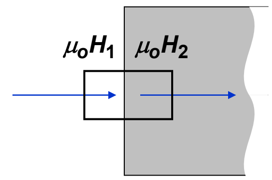
Vector potential of a line-current dipole:

$$A_z(\vec{r}) = \frac{\mu_o I}{2\pi}\ln\left(\frac{r_-}{r_+}\right)$$

Notice the Similarities

Magnetic Field Boundary Conditions - I

The normal component of the B-field at an interface is always continuous



Maxwell equation: $\nabla . \vec{B} = \nabla . \mu_o \vec{H} = 0$

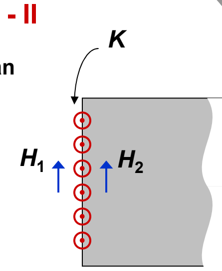
The net magnetic flux coming into a closed surface must equal the magnetic flux coming out of that closed surface (since there are no magnetic charges to generate or terminate magnetic field lines)

Therefore:

$$\mu_o H_2 = \mu_o H_1$$

Magnetic Field Boundary Conditions - II

The discontinuity of the parallel component of the H-field at an interface is related to the surface current density (units: Amps/m) at the interface

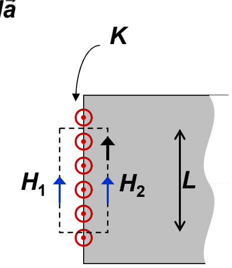


$$H_2 - H_1 = K$$

This follows from Ampere's law:

$$\nabla \times \vec{H} = \vec{J} \quad \text{or} \quad \oint \vec{H} . d\vec{s} = \iint \vec{J} . d\vec{a}$$

The line integral of the magnetic field over a closed contour must equal the total current flowing through the contour



$$H_2 L - H_1 L = K L$$
$$\Rightarrow \quad H_2 - H_1 = K$$

Perfect Metals and Magnetic Fields - I

A perfect metal can have no time varying H-fields inside it

Note: Recall that in magnetoquasistatics one can have time varying H-fields (as long as the time variation is slow enough to satisfy the quasistatic conditions)

The argument goes in two steps as follows:

• **A time varying H-field implies an E-field (from the third equation of magnetoquasistatics)**

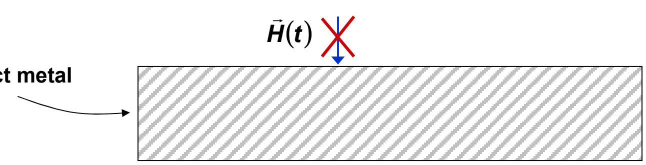
$$\nabla . \mu_o \vec{H}(\vec{r},t) = 0 \qquad \nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) \qquad \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \mu_o \vec{H}(\vec{r},t)}{\partial t}$$

• **Since a perfect metal cannot have any E-fields inside it (time varying or otherwise), a perfect metal cannot have any time varying H-fields inside it**

Perfect Metals and Magnetic Fields - II

At the surface of a perfect metal there can be no component of a time varying H-field that is normal to the surface

perfect metal



The argument goes as follows:

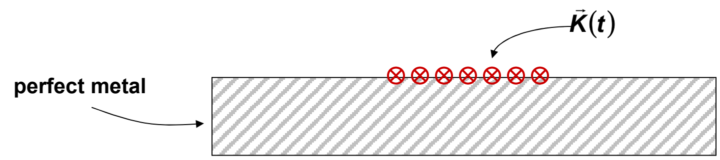
• **The normal component of the H-field is continuous across an interface**

• **So if there is a normal component of a time varying H-field at the surface of a perfect metal there has to be a time varying H-field inside the perfect metal**

• **Since there cannot be any time varying H-fields inside a perfect metal, there cannot be any normal component of a time varying H-field at the surface of a perfect metal**

Perfect Metals and Magnetic Fields - III

Time varying currents can only flow at the surface of a perfect metal but not inside it



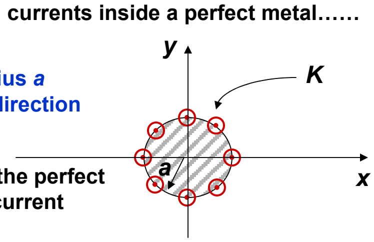
The argument goes as follows:

- Time varying currents produce time varying H-fields
- So if there are time varying currents inside a perfect metal, there will be time varying H-fields inside a perfect metal
- Since there cannot be any time varying H-fields inside a perfect metal, there cannot be any time varying currents inside a perfect metal

Current Flow and Surface Current Density

So how does a (time varying) current flow in perfect metal wires?
Remember there cannot be any time varying currents inside a perfect metal……

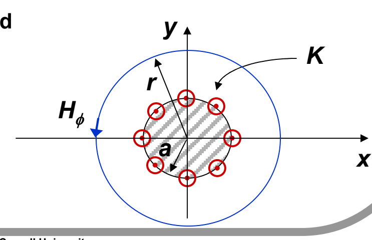
Consider an infinitely long metal wire of radius *a* carrying a (time varying) current *I* in the +z-direction



The current flows entirely on the surface of the perfect metal wire in the form of a uniform surface current density *K*(*t*)

$$K(t) = \frac{I(t)}{2\pi a}$$

Can use Ampere's law to calculate the H-field outside a perfect metal wire carrying a (time varying) current *I*



$$(2\pi r) H_\phi(t) = (2\pi a) K(t)$$

$$\Rightarrow \quad H_\phi(t) = K(t)\frac{a}{r} = \frac{I(t)}{2\pi r}$$

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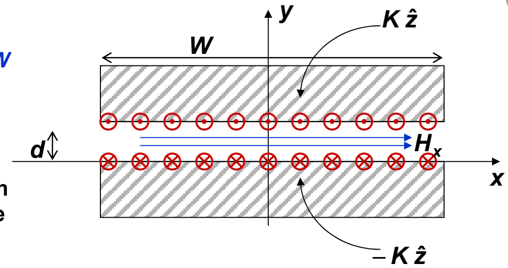
Parallel Plate Conductors

Consider two infinitely long (in the *z*-direction) metal plates, of width *W* and separated by a distance *d*, as shown



The top plate carries a (time varying) current *I* in the +*z*-direction and the bottom plate carries a (time varying) current *I* in the –*z*-direction

Surface current density on the top plate = $K = \dfrac{I}{W}$

If *W* >> *d*, then the H-field inside the plates in very uniform and can be calculated by using the boundary condition:

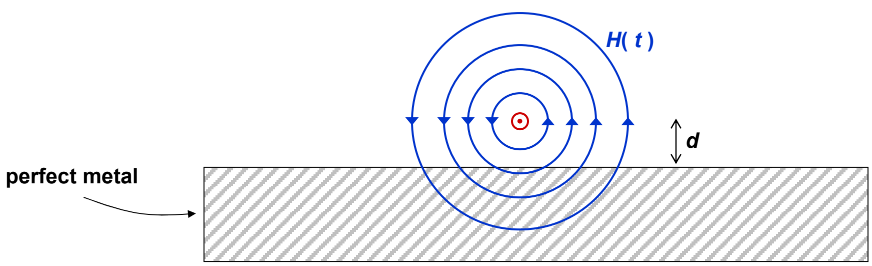
$$H_x\big|_{outside\ metal} - \cancel{H_x\big|_{inside\ metal}}^{\,0} = K$$

$$\Rightarrow \quad H_x\big|_{outside\ metal} = K = \frac{I}{W}$$

One can also use Amperes law directly - see if you can identify an appropriate contour for using Ampere's law to get the same answer

Image Currents - I

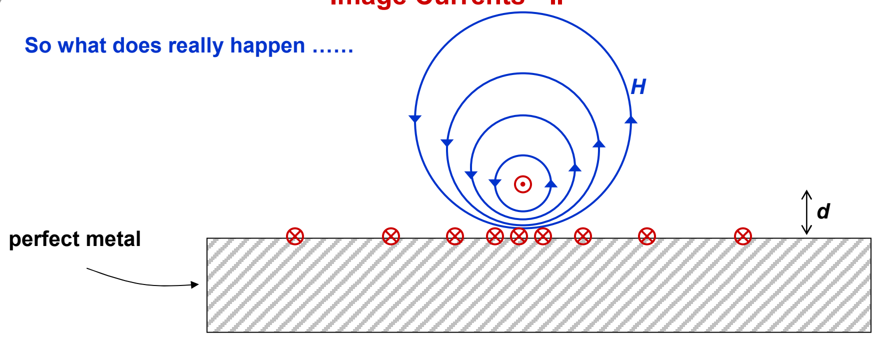
Consider a perfect metal with a wire carrying a time varying current *I*(*t*) in the +z-direction at a distance *d* above the perfect metal, as shown below



Surely this picture cannot be right............there is time varying H-field inside the perfect metal

Image Currents - II

So what does really happen ……



Currents flow on the surface of the perfect metal that completely cancel the time varying H-field inside the perfect metal

In other words, surface currents screen out the time varying H-field from the perfect metal

Question: Is there a better way to understand what the resulting H-field looks like outside the perfect metal?

Image Currents - III

The magnetic field outside the perfect metal can be obtained by imagining a fictitious current element that is a mirror image of the actual current element but carrying a current in the opposite direction

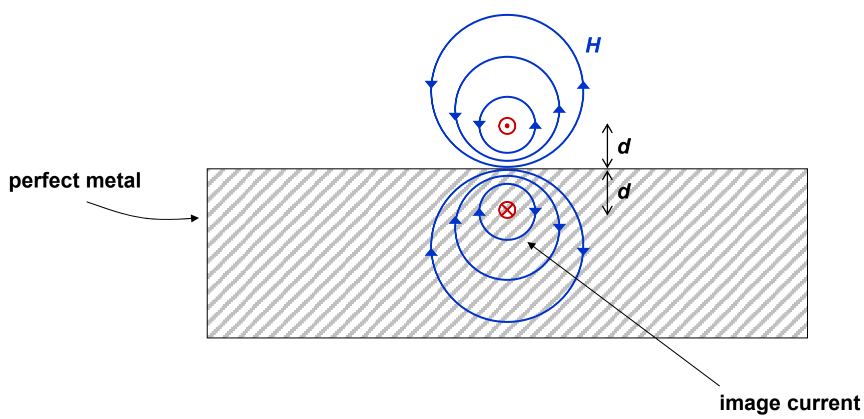
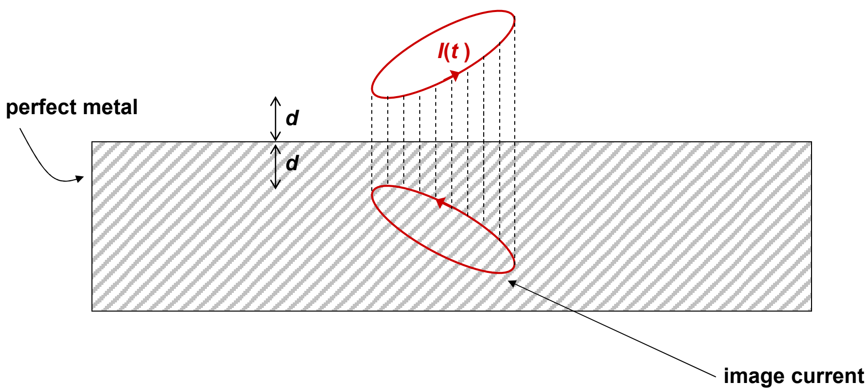


Image Currents - IV

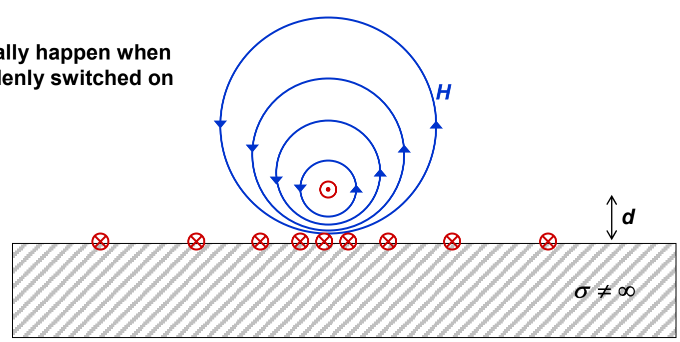
Example: A current loop carrying a time varying current over a perfect metal



Not So Perfect Metals - I

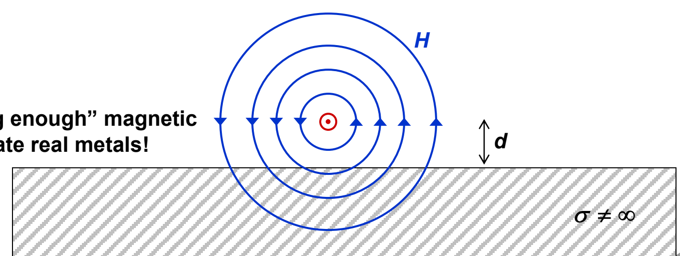
So what does really happen when a current is suddenly switched on at time t=0

Time = t = 0



Time = t = ∞

If you wait "long enough" magnetic field will penetrate real metals!



Not So Perfect Metals – Magnetic Diffusion

Question: How long does it take for the magnetic field to penetrate into real metals?

Answer: Start from these magnetoquasistatic equations:

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \times \vec{E} = -\frac{\partial \mu_o \vec{H}}{\partial t}$$

$$\longrightarrow \vec{J} = \sigma \vec{E}$$

$$\nabla \times \nabla \times \vec{H} = \nabla \times \vec{J} = \sigma \nabla \times \vec{E} = -\sigma \mu_o \frac{\partial \vec{H}}{\partial t}$$

$$\Rightarrow \quad \nabla\left(\nabla . \vec{H}\right) - \nabla^2 \vec{H} = -\sigma \mu_o \frac{\partial \vec{H}}{\partial t}$$

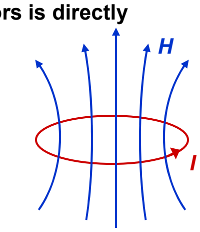
$$\Rightarrow \quad \nabla^2 \vec{H} = \sigma \mu_o \frac{\partial \vec{H}}{\partial t} \longrightarrow$$ **Magnetic diffusion equation**

In time "t" the magnetic field will diffuse a distance "d" into the metal, where:

$$t \approx \frac{\sigma \mu_o d^2}{2}$$

Inductance

The magnetic flux enclosed by current carrying conductors is directly proportional to the current carried by the conductors



$$\text{Magnetic Flux} = \lambda = \iint \vec{B} . d\vec{a} = \iint \mu_o \vec{H} . d\vec{a}$$

$$\lambda \propto I$$

The constant of proportionality is called the **Inductance (units: Henry)**

$$\lambda = L\, I$$

or

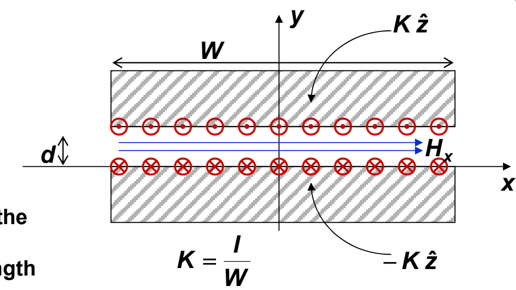
$$L = \frac{\lambda}{I}$$

More accurately: $L = \frac{d\lambda}{dI}$

Inductance of the Parallel Plate Conductors

From previous slide:

$$H_x = K = \frac{I}{W}$$



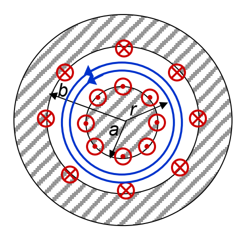
Since the structure is infinite in the *z*-direction, one can only talk about the inductance per unit length (units: Henry/m) (unit length means unit length in the *z*-direction)

$$L = \text{Inductance per unit length} = \frac{\text{Flux per unit length}}{\text{Total current}}$$

$$L = \frac{\mu_o H_x\, d}{I} = \frac{\mu_o\, d}{W}$$

Inductance of the Concentric Cylinders

Consider two perfect metal concentric cylinders infinitely long in the z-direction. The inner one carries a total (time varying) current *I* in the +*z*-direction. The outer shell carries a total (time varying) current *I* in the -*z*-direction



The magnetic field, by symmetry, has only a ϕ-component

Using Ampere's law: $(2\pi\, r) H_\phi(r) = I$

$$\Rightarrow \quad H_\phi(r) = \frac{I}{2\pi\, r}$$

Since the structure is infinite in the *z*-direction, one can only talk about the inductance per unit length (units: Henry/m)

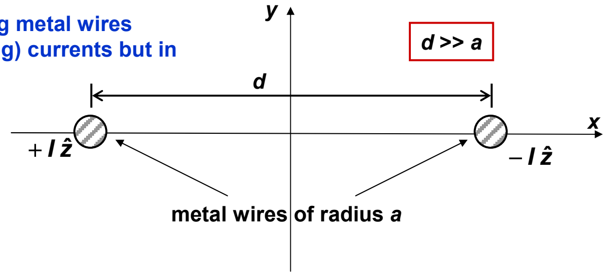
$$L = \frac{\lambda}{I} = \frac{\mu_o \int_a^b H_\phi(r)\, dr}{I}$$

$$= \frac{\mu_o}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance of Two Metal Wires - I

Consider two infinitely long metal wires carrying equal (time varying) currents but in opposite directions

Need to solve:



As long as *d* >> *a*, the surface current density on each metal wire is circularly symmetric and uniform, and so the outside field appears as if one had line-currents at the center of each metal wire

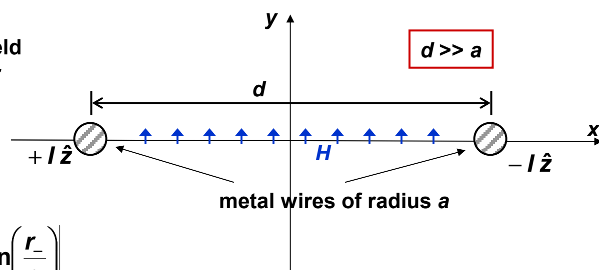
So the vector potential can be written as: $A_z(\vec{r}) = \frac{\mu_o I}{2\pi} \ln\left(\frac{r_-}{r_+}\right)$

Now remember the golden formula from a previous lecture for the magnetic flux:

$$\lambda = \iint \vec{B}.d\vec{a} = \oint \vec{A}\,.\,d\vec{s}$$

Inductance of Two Metal Wires - II

The line integral of the A-field along the indicated contour gives the net flux passing through the contour



$$\lambda = \iint \vec{B}.d\vec{a} = \oint \vec{A}\,.\,d\vec{s}$$

$$= \frac{\mu_o I}{2\pi} \ln\left(\frac{r_-}{r_+}\right)\Bigg|_{\substack{r_-=d \\ r_+=a}} - \frac{\mu_o I}{2\pi} \ln\left(\frac{r_-}{r_+}\right)\Bigg|_{\substack{r_-=a \\ r_+=d}}$$

$$= \frac{\mu_o I}{\pi} \ln\left(\frac{d}{a}\right)$$

$$L = \frac{\lambda}{I} = \frac{\mu_o}{\pi} \ln\left(\frac{d}{a}\right)$$

Looking from the top……..