Preface to the Second Edition

This text is a response to departments of mathematics (many at engineering colleges) that have asked for a combined course in differential equations and linear algebra. It differs from other combined texts in its effort to stress the modern qualitative approach to differential equations, and to merge the disciplines more effectively.

The advantage of combining the topics of linear algebra and differential equations is that the linear algebra provides the underlying mathematical structure, and differential equations supply examples of function spaces in a natural fashion. In a typical linear algebra course, students ask frequently why vector spaces other than $\mathbf{R}^n$ are of interest. In a combined course based on this text, the two topics are interwoven so that solution spaces of homogenous linear systems and solution spaces of homogeneous linear differential equations appear together quite naturally.

Differential Equations

In recent years, the emphasis in differential equations has moved away from the study of closed-form transient solutions to the qualitative analysis of steady-state solutions. Concepts such as equilibrium points and stability have become the focus of attention, diminishing concentration on formulas.

In the past, students of differential equations were generally left with the impression that all differential equations could be "solved," and if given enough time and effort, closed-form expressions involving polynomials, exponentials, trigonometric functions, and so on, could always be found. For students to be left with this impression is a mathematical felony in that even simple-looking equations such as

$$dy/dt = y^2 - t \quad \text{and} \quad dy/dt = e^{ty^2}$$

do not have closed-form solutions. But these equations *do* have *solutions*, which we can see graphically in Figures 1 and 2.

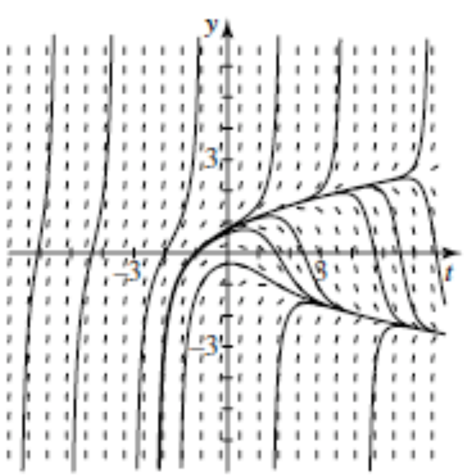

FIGURE 1 $dy/dt = y^2 - t$.

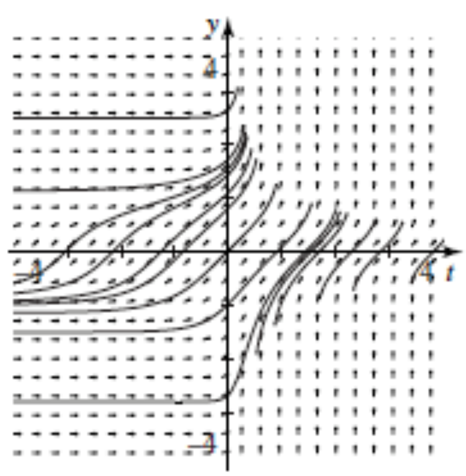

FIGURE 2 $dy/dt = e^{ty^2}$.

In the traditional differential equations course, students spent much of their time grinding out solutions to equations, without real understanding for the\break solutions or the subject. Nowadays, with computers and software packages readily available for finding numerical solutions, plotting vector and directional fields, and carrying out physical simulations, the student can study differential equations on a more sophisticated level than previous students, and ask questions not contemplated by students (or teachers) in the past. Key information is transmitted instantly by visual presentations, especially when students can watch solutions evolve. We use graphics heavily in the text and in the problem sets.

We are *not* discarding the quantitative analysis of differential equations, but rather increasing the qualitative aspects and emphasizing the \ links.

Linear Algebra

The visual approach is especially important in making the connections between differential equations and linear algebra. Although differential equations have long been treated as one of the best applications of linear algebra, in traditional treatments students tended to miss key links. It's a delight to hear those who have taken those old courses gasp with sudden insight when they *see* the role of eigenvectors and eigenvalues in phase portraits.

Throughout the text we stress as a main theme from linear algebra that the general solution of a linear system is the solution to the associated homogeneous equation plus any particular solution.

Consequently, for the first-order linear differential equation we solve the homogeneous equation by separation of variables, and then find a particular solution by a first-order variation of parameters method. Of course, we solve the second-order linear equations and linear systems using the same strategy, giving a more systematic approach to solving linear differential equations, as well as showing how concepts in linear algebra play an important role in differential equations.

Technology Resources

Before we discuss further details of the text, we must sound an alert to the technological support provided, in order that the reader not be
short-changed.

First is the unusual but very effective resource *Interactive Differential Equations* (IDE) that we helped to pioneer. The authors, two of whom were on the development team for IDE, are concerned that IDE might be overlooked by the student and the instructor, although we have taken great pains to incorporate this software in our text presentation. IDE is easily accessed on the internet at

www.aw-bc.com/ide

Interactive Differential Equations (IDE) This original collection of very useful interactive graphics tools was created by Hubert Hohn, at the Massachusetts College of Art, with a mathematical author team of John Cantwell, Jean Marie McDill, Steven Strogatz, and Beverly West, to assist students in really understanding crucial concepts. You will see that a picture (especially a dynamic one that you control) is indeed worth a thousand words. This option greatly enhances the learning of differential equations; we give pointers throughout the text, to give students immediate visual access to concepts.

The 97 "tools" of IDE bring examples and ideas to life. Each has an easy and intuitive point/click interface that requires *no* learning curve. Students have found this software very helpful; instructors often use IDE for short demonstrations that immediately get a point across.

Additional detail is given below under Curriculum Suggestions, and in the section "To the Reader." But keep in mind that IDE is designed as a valuable aid to understanding, and is *not* intended to replace an "open-ended graphic DE solver."

Second, students *must* be able to make their own pictures, with their own equations, to answer their own questions.

We do not want to add computing to the learning load of the students—we would far rather they devote their energy to the mathematics. All that is needed is an ability to draw direction fields and solutions for differential equations, an occasional algebraic curve, and simple spreadsheet capability. A graphing calculator is sufficient for most problems.

A complete computer algebra system (CAS) such as *Derive, Maple, Mathematica*, or *MATLAB* is more than adequate, but not at all necessary. We have found, however, that a dedicated "graphic ODE solver" is the handiest for differential equations, and have provided a good one on the text website:

http://www.math.rice.edu/~dfield/dfpp.html

ODE Software (Dfield and Pplane John Polking of Rice University has created our "open-ended graphic DE solver," originally as a specialized front-end for MATLAB, but now there is a stand-alone Java option on our website. *Dfield* and *Pplane* provide an easy-to-use option for students and avoid the necessity of familiarity or access to a larger computer algebra system (CAS).

Finally, many instructors have expressed interest in projects designed for the more powerful CAS options; we will make a collection available on the text website cited above.

CAS Computer Projects Professor Don Hartig at California Polytechnic State University has designed and written a set of Computer Projects utilizing *Maple* for Chapters 1–9. This package will be added to the text website, to provide a guide to instructors and students who might want to use a computer algebra system (CAS). These can be adapted to another CAS, and other projects may be added.

Differences from Traditional Texts

Although we have more pages explicitly devoted to differential equations than to linear algebra, we have tried to provide all the basics of both that either course syllabus would normally require. But merging two subjects into one (while at the same time enhancing the usual quantitative techniques with qualitative analysis of differential equations) requires streamlining and simplification. The result should serve students well in subsequent courses and applications.

Some Techniques De-Emphasized Many of the specialized techniques used to solve small classes of differential equations are no longer included within the confines of the text, but have been retired to the problem set. The same is true for some of the specialized techniques of linear algebra.

Dynamical Systems Philosophy We focus on the long-term behavior of a system as well as its transient behavior. Direction fields, phase plane analysis, and trajectories of solutions, equilibria, and stability are discussed whenever appropriate.

Exploration Problems for nontraditional topics such as bifurcation and chaos often involve guided or open-ended exploration, rather than application of a formula to arrive at a specific numerical answer. Although this approach is not traditional, it reflects the nature of how mathematics advances.

This experimental stage is the world toward which students are headed; it is essential that they learn how to do it, especially how to organize and communicate about the results.

Problem Sets Each problem set involves most or all of the following:

- traditional problems for hand calculation (and understanding of techniques)
- additional traditional techniques
- graphical exercises (drawing, matching) to gain understanding of different representations
- real world applications
- some open-ended questions or exploration
- suggested journal entries (writing exercises)

Writing in Mathematics In recent years, the "Writing Across the Curriculum" crusade has spread across American colleges and universities, with the idea of learning through writing. We include "Suggested Journal Entries" at the end of each problem set, asking the student to write something about the section. The topics suggested should be considered simply as possible ideas; students may come up with something different on their own that is more relevant to their own thinking and evolving comprehension. Another way to ask students to keep a scholarly journal is to allow five minutes at the end of class for each student to write and outline what he or she does or does not understand. The goal is simply to encourage writing about mathematics; the degree to which it raises student understanding and performance can be amazing! Further background is provided in the section "To the Reader."

Historical Perspective We have tried to give the reader an appreciation of the richness and history of differential equations and linear algebra through footnotes and the use of "Historical Notes," which are included throughout the book. They can also be used by the instructor to foster discussions on the history of mathematics. \vspace*{2pt}

Applications We include traditional applications of differential equations: mechanical vibrations, electrical circuits, biological problems, biological chaos, heat flow problems, compartmental problems, and many more.

Many sections have applications at the end, where an instructor can choose to spend extra time. Furthermore, many problems introduce new applications and ideas not normally found in a beginning differential equations text: reversible systems, adjoint systems, Hamiltonians, and so on, for the more curious reader.

The final two chapters introduce related subjects that suggest ideal follow-up courses.

- Discrete Dynamical Systems: Iterative or difference equations (both linear and nonlinear) have important similarities and differences from differential equations. The ideas are simple, but the results can be surprisingly complicated. Subsections are devoted to the discrete logistic equation and its path to chaos.

  Control Theory: Although one of the most important applications of differential equations is control theory, few books on differential equations spend any time on the subject. This short chapter introduces a few of the important ideas, including feedback control and the Pontryagin maximum principle.

Obviously you will not have time to look at every application—being four authors with different interests and specialties, we do not expect that. But we would suggest that you choose to spend some time on what is closest to *your* heart, and in addition become aware of the wealth of other possibilities.

Changes in the Second Edition

Our goal has been to make the connection between the topics of differential equations and linear algebra even more obvious to the student. For this reason we have emphasized solution spaces (of homogeneous linear algebraic systems in Chapter 3, and homogenous linear differential equations in Chapters 4 and 6), and stressed the use of the Superposition Principle and the Nonhomogeneous Principle in these chapters.

Many of the changes and corrections have been in response to many helpful comments and suggestions by professors who have used the first edition in their courses. We have greatly benefited from their experience and insight.

The following major changes and additions are part of the new edition. We have also trimmed terminology, reorganized, and created a less-cramped layout, to clarify what is important.

Chapter 1: First-Order Differential Equations Introductory Section 1.1 has been rewritten and now includes material on direct and inverse variation. Section 1.2 has been expanded with many more problems specific to the qualitative aspects of solution graphs. Runge-Kutta methods for numerical approximations are part of Section 1.4.

Chapter 2: Linearity and Nonlinearity We have tried to make a clearer presentation of the basic structure of linearity, and have added a few new applications.

Chapter 3: Linear Algebra
This chapter contains several examples of homogenous DE solution spaces in the section on vector spaces. The applications of the
Superposition Principle and Nonhomogeneous Principle have been emphasized throughout. The issue of linear independence has been amplified.

Chapter 4: Higher-Order Linear Differential Equations This chapter emphasizes solution spaces and linear independence of solutions. We have added a new section on variation of parameters.

Chapter 5: Linear Transformations The generalized eigenvector section has been expanded.

Chapter 6: Linear Systems of Differential Equations This chapter has been expanded to include more information on the sketching of phase portraits, particularly for systems with nonreal eigenvalues. Two new sections have been added, on the matrix exponential and on solutions to nonhomogeneous systems of DEs, including the undetermined coefficients and variation of parameters methods (found in Sections 8.1 and 8.2 in the first edition). New examples of applications to circuits, coupled oscillators, and systems of tanks have been added.

Chapter 7: Nonlinear Systems of Differential Equations This is the big payoff that shows the power of all that comes before. This chapter has been extended to include as Section 7.5 the chaos material on forced nonlinear systems (Section~8.5 in the first edition).

Chapter 8: Laplace Transforms This chapter has been reorganized so that it only contains material on Laplace Transforms. The material has been expanded to four sections and a new section on Systems of Laplace Transforms has been added.

Chapter 9: Discrete Dynamical Systems This chapter has been updated to reflect new research by Samer Habre of the Lebanese American University and one of the authors.

Chapter 10: Control Theory and the Appendices These have been slightly revised to be consistent with the improved organization of this edition.

Problems Some 500 problems have been added to the approximately 2,000 problems of the first edition. A problem number in color signifies
that there is a brief answer at the back of the book.

Curriculum Suggestions

Scientific and technological programs at many colleges and universities are increasingly squeezed by new courses required by the rapid advances in their fields. Consequently the amount of time for mathematical support courses is being pinched. Students are expected to acquire more skills in fewer courses. One result of this pinch at a number of institutions has been the appearance of an integrated course in linear algebra and differential equations, with a target audience of students having had a smattering of matrix algebra in precalculus or discrete mathematics and a minimal introduction to differential equations in calculus. It is for these students that this material is designed. The presentation is informal. Theory is presented visually and intuitively. Problem-solving skills are emphasized. Contemporary computer support is integrated into the text and problems.

The text is intended as a sophomore–junior level course in differential equations and linear algebra, for students majoring in science, engineering, and mathematics who have taken a one-year course in calculus. For a one-semester course, most instructors would probably rather concentrate on Chapters 1 through 8, skipping sections such as those on chaos (Sections 7.4–7.5) or Laplace transforms (Sections 8.3 and 8.4) as appropriate for their students. Two semesters would allow for more complete coverage of all the chapters.

The table below suggests a possible selection of materials for a one-semester course.

Basic Text Organization

	Core	Optional	Enrichment
Chapter 1: First-Order Differential Equations	1.1–1.5		
Chapter 2: Linearity and Nonlinearity	2.1–2.5	2.6	
Chapter 3: Linear Algebra	3.1–3.6		
Chapter 4: Second-Order Linear Differential Equations	4.1–4.5	4.6–4.7	
Chapter 5: Linear Transformations	5.1–5.3	5.4	
Chapter 6: Linear Systems Differential Equations	6.1–6.4	6.5–6.7	
Chapter 7: Nonlinear Systems Differential Equations	7.1–7.2	7.3	7.4–7.5
Chapter 8: Laplace Transforms	8.1–8.4	8.5	
Chapter 9: Discrete Dynamical Systems			9.1–9.3
Chapter 10: Control Theory			10.1–10.3

The basic text can be adapted to many different circumstances.

- Students with strong DE backgrounds (e.g., from reformed calculus courses) can use Chapter 1 as an overview and omit or review Sections 2.2–2.5.
- Students with strong matrix or linear algebra backgrounds (e.g., from courses in finite or discrete mathematics) will be able to omit or simply review Sections 3.1–3.4.
- Courses emphasizing linear algebra should add Appendix LT to the core, and could include Chapter 9.
- Courses emphasizing differential equations with Laplace transforms would include Chapter 8 and should consider Chapter 10.
- Courses emphasizing differential equations with dynamical systems instead of Laplace transforms would cover Chapter 9.

•A pure differential equations course could omit Sections 3.3–3.6, and all of Chapter 5 except Section 5.3, replacing them with additional sections from Chapters 7 and 8, plus Chapter 10.

Notes on the Uses of IDE

Instructors might find useful the following notes from co-author J. M. McDill on her use of the *Interactive Differential Equations* (IDE) software during a differential equations and linear algebra course at California State Polytechnic University at San Luis Obispo. IDE is available at this website:

http://www.aw-bc.com/ide

Our course is over one ten-week quarter that meets four days per week. It is very crowded with material. I use a variety of IDE interactive illustrations; however, I use four or five basic labs to help me make it through the material in time.

• On the day following the introduction of the general method for solutions of linear homogeneous equations with constant coefficients, I show the substitution for an unforced mass-spring system and assign *Lab 9: Linear Oscillators: Free Response*. In this lab the students can vary each of the parameters, investigate the cases for the damped and undamped mass—spring systems, and obtain concrete visual examples of the undamped, underdamped, critically damped, and overdamped cases.

• The day after I introduce the method of undetermined coefficients for nonhomogeneous linear differential equations with constant coefficients, I show the substitution of parameters for a forced mass-spring system and assign *Lab 12: Forced Vibrations*: An Introduction. In this lab the students can vary parameters and observe that the motion of the mass is the sum of the transient and steady-state solutions by means of the software, which links graphs to a model of the mass-spring system. (I always found the presentation of the graphical material at the blackboard to be time-consuming.) Also, the undamped forced mass-spring system can be investigated by the students. They can assign parameters to produce pure resonance and beats. Even if formal lab time is not available to the students, IDE can be used as a lecture demonstration and/or the appropriate labs can be assigned for homework.

• As a further illustration of the usefulness of the analytical techniques for solving linear differential equations with constant coefficients, I emphasize the analogy between series circuits and mass-spring systems. I assign *Lab 13: Electrical Circuits* as homework to illustrate this analogous behavior.

• By the time we approach the materials in Chapter 6, I am feeling somewhat rushed. I cover Sections 6.1–6.3. Then I assign *Lab 16: Linear Classification* as homework (usually over a weekend) before I lecture on Section 6.4: Stability and Linear Classification. I find that students who have covered the~material visually as guided by the lab do not find the lecture so mind-boggling.

• I usually also sneak in *Lab 18: Romeo and Juliet*, as I introduce the phase plane. I find that it exerts a certain fascination for the students and provides an unforgettable if unconventional connection between the phase plane trajectories and the solution graphs.