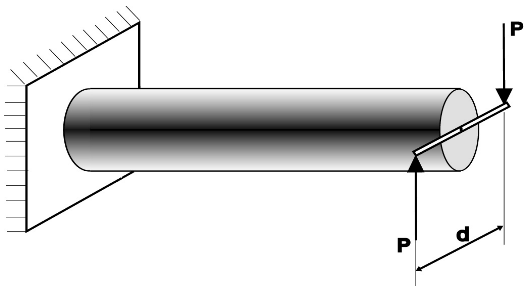
Theoretical and Applied Mechanics

Virtual Labs, Real Data

Basic Theory of Torsion

CORNELL

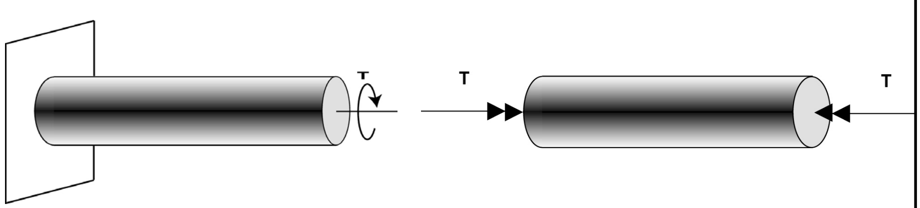
Torsion refers to the twisting of a shaft loaded by torque, or twisting couples.



In this example, the magnitude of the moment, or torque, *T*, due to the couple is

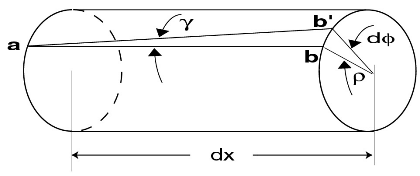
$$T = Pd$$

Alternate representations of torque



Use the right hand rule to understand the above. Torque is a moment with its vector direction along the axis of the shaft. The torque applied in these figures is in the same direction as in slide 1.

Deformation due to torsion



Note: Relations here are based solely on geometry and so they are valid for circular shaft of any material, linear or non-linear, elastic or non-elastic.

- ρ is the radial distance to any point.
- ϕ is the angle of twist in radians
- γ is the shear strain.
- The horizontal line ab moves to ab'
- The shear strain is $\gamma = \frac{bb'}{ab}$
- From geometry $\gamma dx = \rho d\phi$
- So the strain $\gamma = \rho \frac{d\phi}{dx}$
- If shaft has uniform section, length *L*,
  $$\phi = L\frac{d\phi}{dx}$$
- Hence twist and strain are related as
  $$\gamma = \frac{\rho\phi}{L}$$

Stresses in torsion

•We use Hooke's law for a linear elastic material

$$\tau = G\gamma$$

•From slide 3, the shear strain is

$$\gamma = \frac{\rho\phi}{L}$$

•Thus the stress is

$$\tau = G\frac{\rho\phi}{L}$$

•The torque, *T*, is found by integrating the stress*distance over the cross section, *S*, of the shaft

$$T = \int_S \tau \rho \, dA$$

Relation of torque and twist

- Using stress from previous relations, we get

$$T = \int_S G\frac{\phi}{L}\rho^2 dA = G\frac{\phi}{L}\int_S \rho^2 dA = G\frac{\phi}{L}J$$

where *J* is the polar moment of inertia

- Using the above we find the relation between the twist and the torque

$$\phi = \frac{TL}{GJ}$$

- And the we can write the shear stress as

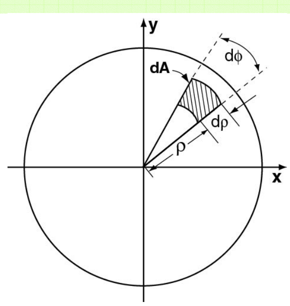
$$\tau = G\frac{\rho\phi}{L} = G\frac{\rho}{L}\frac{TL}{JG} = \frac{T\rho}{J}$$

Polar Moment of Inertia

Definition: The Polar Moment of Inertia is defined as the integral

$$J = \int_S \rho^2 dA$$

If 'O' is the centroid of the area, then ρ is the distance from the point 'O' to the element of area dA.



*Solid circular cross section:

$$J = \frac{\pi r^4}{2} \qquad \{r = radius\}$$

*Hollow Circular Cross Section:

$$J = \frac{\pi(r_o^4 - r_i^4)}{2}$$

where, $\left\{\begin{matrix} r_o = & outer & radius \\ r_i = & inner & radius \end{matrix}\right\}$

Torsional stiffness

Think of the shaft as a torsional spring with

$$T = k\phi$$

k is the stiffness of the torsional spring.

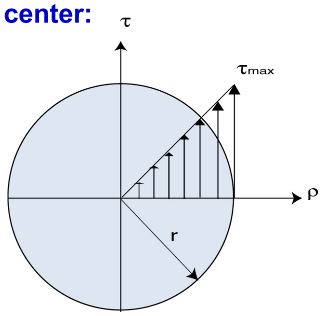
Since $$T = \frac{GJ}{L}\phi$$

Torsional stiffness : $k = \dfrac{GJ}{L}$

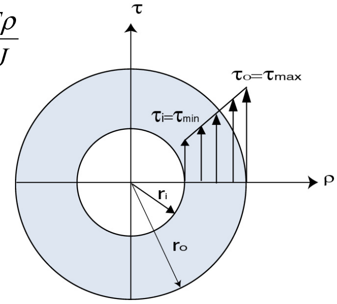
Stress distribution in a shaft.

In a circular shaft shear stress varies linearly from center:

$$\tau = \frac{T\rho}{J}$$



Solid Shaft

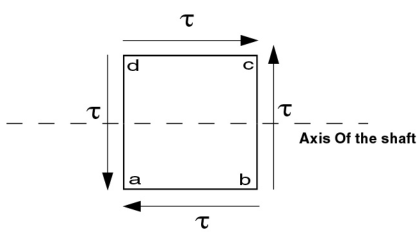


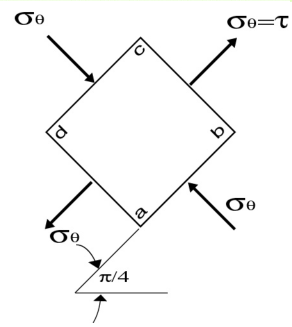
Hollow Shaft

Note: The shear stress is maximum for the outermost element where the radii is maximum.

Behavior of brittle and ductile materials under torsion

The state of pure shear is equivalent to a state of pure compressive and tensile stresses for an element rotated through 45^0.



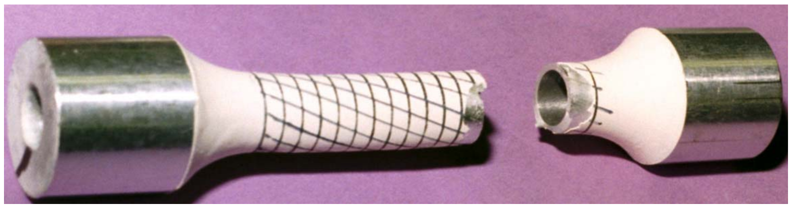


If material is weak in tension, it will fail along a plane 45^0 to the longitudinal axis of the shaft. If weak in shear it will fail along a plane 90^0 to the axis of the shaft

Failure of a brittle material under torsional loading

Failure occurs in tension along a helix inclined at an angle of 45^0 to the longitudinal axis of the femur.

Failure of a ductile material under torsional load



Failure occurs in shear along a plane perpendicular to the longitudinal axis.

Elastic-plastic torsion

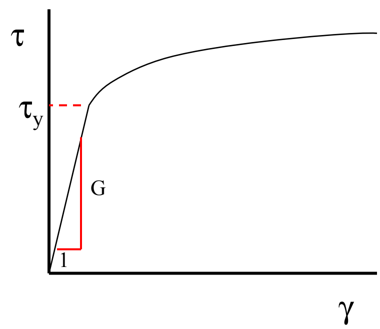
- **Hooke's law does not apply when shear stress exceeds shear yield strength, τ_y.**
- **Torque at start of yield is given by**

$$T_{yield} = \tau_y \frac{J}{r_o}$$

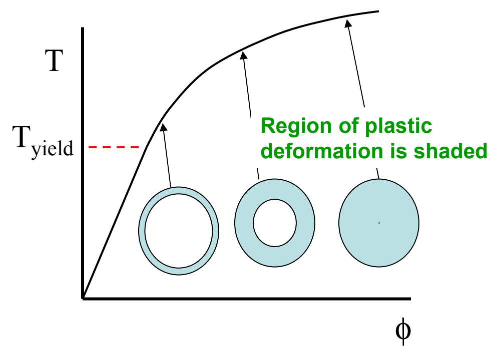
where r_o *is outer radius of shaft.*

- **Outer surface yields first.**
- **As torque increases, region of yielding expands to cover entire cross section.**

Elastic-plastic torsion example



Schematic shear stress-strain curve



Schematic torque-twist curve

•As deformation progresses, region of plastic deformation, $\tau > \tau_y$, spreads towards center of shaft

Summary of topics covered

- Torsion defined
- Strain due to torsion
- Relationship between strain, twist and torque
- Polar moment of inertia
- Torsional stiffness
- Stress distribution in a shaft
- Failure of ductile and brittle materials
- Elastic-plastic torsion

Summary of equations

$$\gamma = \frac{\rho\phi}{L}$$

$$\tau = \frac{T\rho}{J} \qquad \tau_{\max} = \frac{Tr_O}{J}$$

$$T = k\phi$$

$$k = \frac{GJ}{L}$$

$$J = \frac{\pi(r_o^4 - r_i^4)}{2}$$

Slide 1

In this presentation we will define what torque and torsion are and we will derive the theory of torsion of circular shafts. Implications of torsion theory to the failure of materials is also discussed. Torsion refers to the twisting of a shaft loaded by torque, or twisting couples**.** For example, in the generation of electricity shafts carry torque from the turbine to the generator. An example of torsional loading is shown here. In this example we load the shaft by two equal and opposite forces acting on a bar perpendicular to the shaft. The moment generated by these forces is sometimes called a couple. The magnitude of the moment due to this couple is given by **P times d,** where, **P** is the applied force and **d** is the distance between the lines of action of the forces.This twisting couple is also called the 'Torque' or 'Twisting Moment'.

Slide 2

Two alternate ways of depicting torque are shown here. In the left-hand figure the torque is shown as a loop with an arrow depicting its direction. In the right-hand figure the torque is shown as a vector moment. The direction of the moment is parallel to the shaft. The sign of the moment, can be understood using the right hand rule. The rule is that if you rotate your right hand in the direction of the applied torque, then your thumb points in the direction of the vector indicated by two arrowheads in the right-hand figure.

Slide 3

We will now review the derivation and interpretation of the theory of torsion of circular shafts. We start by looking at a small section of length dx of a circular shaft under torsion. During twisting, one end of the shaft will rotate about the longitudinal axis with respect to the other end . The magnitude of this rotation is measured in terms of the angle in radians by which one end rotates relative to the other. This is called the **'Angle of Twist'**. It can be seen that the line ab, which was initially horizontal, rotates through an angle gamma, and moves to the line ab'. Here $d\phi$ is the angle of twist.

The shear strain, gamma is the angle between ab and ab'. It is found by the distance bb' divided by the distance ab. Using geometry, the arc length $\rho d\phi = \gamma dx$. Thus we can write the strain as

$\gamma = \rho \dfrac{d\phi}{dx}$. Let's assume that we are dealing with a shaft of uniform cross section and materials, thus the total twist, ϕ over a length L is simply $\phi = L\dfrac{d\phi}{dx}$. Combining the third and fourth equations we get the final equation, giving the relation of shear strain to twist (ϕ), radial distance (ρ), and shaft length (L). Note that all the relations here, are based solely on the geometry of the circular shaft. Hence they are valid for any type of material. This is not so in what follows, the calculation of stresses based on linear elastic material behavior.

Slide 4

For a linear elastic material, using Hooke's law, we can write the shear stress as $\tau = G\gamma$, where, G is the Shear Modulus. The shear strain on a small area of material situated at a distance ρ from the center, was found in slide 3 to be: $\gamma = \dfrac{\rho\phi}{L}$. Thus, using Hooke's law, as $\tau = \dfrac{G\rho\phi}{L}$. The torque, **T**, is calculated by integrating over the cross section the product of shear stress, τ, and the distance, ρ, from the center of the shaft .

Slide 5

Substituting the stress from previous expressions, we find that torque is the integral of $\int_S G\dfrac{\phi}{L}\rho^2 dA$ over the cross section of the shaft. Pulling out the terms that do not vary over the cross section we get that $T = G\dfrac{\phi}{L}J$, where J is the polar moment of inertia and is defined as $J = \int_S \rho^2 dA$. We will discuss polar moment of inertia, J, on the next slide. Rearranging the terms, we can write the angle of twist, ϕ, as $\phi = \dfrac{TL}{GJ}$. We can also find the stress from $\tau = G\dfrac{\rho\phi}{L}$, and then substituting for ϕ to get $\tau = \dfrac{T\rho}{J}$.

Slide 6

The moment of inertia about an axis perpendicular to the plane of an area is called the Polar Moment of Inertia. If dA is the area of a small element at a distance ρ from the center of the cross section, then the Polar Moment of Inertia, J, is defined as the integral over the cross section of the product of distance squared and the small area dA .For a solid circular shaft, the polar moment of inertia is given by $J = \int_S \rho^2 dA$. Similarly we can write the expression for a hollow shaft.

Slide 7

To understand what torsional stiffness is, we can think of the shaft as a torsional spring with torque equal to a spring constant, k, times rotation ϕ. From previous relations, we already know that $T = \frac{GJ\phi}{L}$. Comparing these two relations, we find that the torsional stiffness, k, is $k = \frac{GJ}{L}$.The stiffness increases with increasing shear modulus, increases with the fourth power of the shaft diameter since J is proportional to r to the fourth power, and decreases as the shaft gets longer.

Slide 8

Now let's use the theory to examine in further detail the shear stress distribution in a shaft under torsional loading. We already know that the shear stresses are directly proportional to the distance from the center. It means that, for any circular shaft, the shear stress would be maximum for an element which is farthest from the center. The figure on the left shows the shear stress distribution in a solid shaft. The shear stress is zero at the center while it is the maximum at the surface. The figure on the right shows the distribution of shear stress for a hollow shaft. The shear stress is minimum for on the inside surface and maximum on the outer surface. As the thickness of the wall of the shaft decreases relative to the shaft diameter, the difference between the stress on the inside and outside of the shaft decreases and you obtain a more uniform stress field.

Slide 9

We might ask the question "Do all the materials fail in a similar manner when subjected to torsional Loading?" As you will see the answer is no. To understand this you will remember that whenever a structure is subjected to torsion, the elements are in a state of pure shear. By considering the transformation of stresses, this state of pure shear, is equivalent to a state of pure compression and pure tension on planes rotated forty five degrees to the axis of the shaft. One implication of this stress state is that if the material is weak in tension, it will fail along a plane forty five degrees to the longitudinal axis of the shaft. However, if the material is ductile, meaning that it can deform in shear before breaking, it may fail along a plane ninety to the axis of the shaft.

Slide 10

In this picture we see a turkey tibiotarsus bone fractured under torsional loading. It can be seen that failure occurs along a helix inclined at an angle of 45^0 to the longitudinal axis. This type of failure is typical of brittle materials since they tend to fail along planes perpendicular to the direction of maximum tension.

Slide 11

This picture shows a ductile metal shaft fracture in torsion. It has failed by shearing off along a plane perpendicular to the axis of the shaft. This type of failure is typical to ductile materials since they can deform in shear more readily than they can fracture in tension.

Slide 12

We now look at the behavior of a shaft made of a ductile material under torsional loading. Hooke's law applies only when the shear stress is below the yield stress, τ_y. The value of the torque at the start of yield is given by $T_{yield} = \frac{\tau_y J}{r_0}$, where τ_y is the shear yield strength of the material and r_0 is the outer radius of the shaft. Note also that the outer surface yields first and then the plastic deformation progresses to the center of the shaft as the torque increases.

Slide 13

Shown on the left is a typical stress versus strain curve for a ductile material. The slope of the line in the linear elastic region is the shear modulus for the material. The schematic on the right is a plot of torque versus rotation. This curve is similar to the stress-strain curve. The plot shows the evolution of plastic deformation for the shaft as the torque increases. Just at the end of the linear portion of the plot, the outer portion of the shaft begins to yield. As torque increases, the plastic zone expands to the inner core.

Slide 14

We have reviewed a number of topics in the theory of torsion of circular shafts. Torsion refers to loading of a shaft by a moment parallel to the shaft. This moment is called a torque. The strain in torsion is found by geometric considerations. Knowing strain we found stress using Hooke's law. Integrating the stress over the cross sectional area of the shaft, we derived the relation between shear strain, twist and torque. This relation involves the polar moment of inertia. Thinking of the shaft as a torsional spring we found the spring constant in terms of the geometry and shear modulus of the shaft. Stresses in solid and hollow shafts are discussed, and brittle and ductile failure explored.

Slide 15

The basic equations of torsion of circular shafts are given here for reference. You may want to refer to them as you work on your analyses and reports. Of course, you can also find these results in your mechanics of materials textbook.