Revealed Preferences in a Heterogeneous Population (Online Appendix)

Stefan Hoderlein* Jörg Stoye†

Boston College Cornell University

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*Stefan Hoderlein, Department of Economics, Boston College, 140 Commonwealth Avenue, Chestnut Hill, MA 02467, USA, email: stefan_hoderlein@yahoo.com.

†Jörg Stoye, Department of Economics, Cornell University, Uris Hall, Ithaca, NY 14853, stoye@cornell.edu.

Appendix I - Summary Statistics of Data: Household Characteristics, Income and Normalized Expenditures

Variable	Minimum	1st Quartile	Median	Mean	3rd Quartile	maximum
number of female	0	1	1	1.073	1	2
number of retired	0	0	0	0.051	0	1
number of earners	0	1	2	1.692	2	2
Age of HHhead	19	31	49	46	58	90
Fridge	0	1	1	0.987	1	1
Washing Machine	0	1	1	0.882	1	1
Centr. Heating	0	1	1	0.804	1	1
TV	0	1	1	0.874	1	1
Video	0	0	0	0.407	1	1
PC	0	0	0	0.792	0	1
number of cars	0	1	1	1.351	2	10
number of rooms	1	4	5	5.455	6	26
HHincome	6.653	37.550	52.210	61.820	73.920	3981.000
Food	0	5.565	7.346	7.867	9.602	52.519
Housing	0	4.052	7.859	9.715	12.910	375.486
Energy	0	1.271	1.812	2.121	2.509	34.103

Appendix II - Proofs

Proposition 1 Recall the Fréchet-Hoeffding bounds: For any two random variables X_1 and X_2 and events (in the relevant algebras) A_1 and A_2, one has the tight bounds

$$\max\{\mathbb{P}(X_1 \in A_1) + \mathbb{P}(X_2 \in A_2) - 1, 0\} \leq \mathbb{P}(X_1 \in A_1, X_2 \in A_2) \leq \min\{\mathbb{P}(X_1 \in A_1), \mathbb{P}(X_2 \in A_2)\}.$$

To see validity of the lower bound, note that

$$\begin{aligned}
& \mathbb{P}_{yz}(\text{WARP violated}) \\
= \ & \mathbb{P}_{yz}(Q_s \leq (y_t - y_s)/(p_t - p_s), Q_t \geq (y_t - y_s)/(p_t - p_s), Q_s \neq Q_t) \\
= \ & \mathbb{P}_{yz}(Q_s \leq (y_t - y_s)/(p_t - p_s), Q_t \geq (y_t - y_s)/(p_t - p_s)) - \\
& \mathbb{P}_{yz}(Q_s \leq (y_t - y_s)/(p_t - p_s), Q_t \geq (y_t - y_s)/(p_t - p_s), Q_s = Q_t) \\
= \ & \mathbb{P}_{yz}(Q_s \leq (y_t - y_s)/(p_t - p_s), Q_t \geq (y_t - y_s)/(p_t - p_s)) - \mathbb{P}_{yz}(Q_s = Q_t = (y_t - y_s)/(p_t - p_s)) \\
\geq \ & \max\left\{\mathbb{P}_{yz}(Q_s \leq (y_t - y_s)/(p_t - p_s)) - \mathbb{P}_{yz}(Q_t < (y_t - y_s)/(p_t - p_s)), 0\right\} - \\
& \min\left\{\mathbb{P}_{yz}(Q_s = (y_t - y_s)/(p_t - p_s)), \mathbb{P}_{yz}(Q_t = (y_t - y_s)/(p_t - p_s))\right\},
\end{aligned}$$

where the first equality spells out the event that WARP is violated, the next two steps use basic probability calculus, and the last step uses the *lower* Fréchet-Hoeffding bound on $\mathbb{P}_{yz}(Q_s \leq (y_t - y_s)/(p_t - p_s)), Q_t \geq (y_t - y_s)/(p_t - p_s))$ as well as the *upper* Fréchet-Hoeffding bound on $\mathbb{P}_{yz}(Q_s = Q_t = (y_t - y_s)/(p_t - p_s))$. The expression in the lemma is generated by taking the maximum between the last expression and zero, observing that this renders redundant the max-operator in the preceding display.

The bound is tight because a joint distribution of (Q_s, Q_t) that achieves it can be constructed as follows: First, assign probability $\min\left\{\mathbb{P}_{yz}(Q_s = (y_t - y_s)/(p_t - p_s)), \mathbb{P}_{yz}(Q_t = (y_t - y_s)/(p_t - p_s))\right\}$ to the event $(Q_s = Q_t = (y_t - y_s)/(p_t - p_s))$. Second, remove this probability mass from the marginal distributions of (Q_s, Q_t) and rescale them so they integrate to 1. Third, the joint distribution of $(Q_s, Q_t | Q_s = Q_t = (y_t - y_s)/(p_t - p_s)$ does not hold) is characterized by those rescaled marginal distributions and the Fréchet-Hoeffding lower bound (perfectly positive dependence) copula.

To see validity of the upper bound, note that

$$\mathbb{P}_{yz}(\text{WARP violated}) \leq \min\left\{\mathbb{P}_{yz}(Q_s \leq (y_t - y_s)/(p_t - p_s)), \mathbb{P}_{yz}(Q_t \geq (y_t - y_s)/(p_t - p_s))\right\}$$

by the upper Fréchet-Hoeffding bounds and furthermore that

$$\begin{aligned}
& \mathbb{P}_{yz}(\text{WARP violated}) \\
= \; & \mathbb{P}_{yz}((Q_s \leq (y_t - y_s)/(p_t - p_s), Q_t > (y_t - y_s)/(p_t - p_s)) \\
& \text{or } (Q_s < (y_t - y_s)/(p_t - p_s), Q_t \geq (y_t - y_s)/(p_t - p_s))) \\
\leq \; & 1 - \mathbb{P}_{yz}(Q_s \geq (y_t - y_s)/(p_t - p_s), Q_t \leq (y_t - y_s)/(p_t - p_s)) \\
\leq \; & 1 - \min\{\mathbb{P}_{yz}(Q_s \geq (y_t - y_s)/(p_t - p_s)), \mathbb{P}_{yz}(Q_t \leq (y_t - y_s)/(p_t - p_s))\} \\
= \; & \max\{\mathbb{P}_{yz}(Q_s < (y_t - y_s)/(p_t - p_s)), \mathbb{P}_{yz}(Q_t > (y_t - y_s)/(p_t - p_s))\},
\end{aligned}$$

where all equalities and the first inequality use basic probability calculus and the second inequality utilizes a lower Fréchet-Hoeffding bound on $\mathbb{P}_{yz}(Q_s \geq (y_t - y_s)/(p_t - p_s), Q_t \leq (y_t - y_s)/(p_t - p_s))$.

To see that the bound is tight, note that it is achieved by the Fréchet-Hoeffding upper bound (perfectly negative dependence) copula. Intermediate values can be attained by mixing the two distributions that achieve the bounds.

Lemma 2.2 Assume positive quadrant dependence. Recall that $\underline{\mathcal{B}}_{s,t}$ is a lower contour set and $\underline{\mathcal{B}}_{t,s}$ an upper one, hence

$$\mathbb{P}_{yz}(\mathbf{Q}_s \notin \underline{\mathcal{B}}_{s,t}, \mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}) \geq \mathbb{P}_{yz}(\mathbf{Q}_s \notin \underline{\mathcal{B}}_{s,t})\mathbb{P}_{yz}(\mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}),$$

hence

$$\begin{aligned}
\mathbb{P}_{yz}(\mathbf{Q}_s \in \underline{\mathcal{B}}_{s,t}, \mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}) &= \mathbb{P}_{yz}(\mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}) - \mathbb{P}_{yz}(\mathbf{Q}_s \notin \underline{\mathcal{B}}_{s,t}, \mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}) \\
&\leq \mathbb{P}_{yz}(\mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}) - \mathbb{P}_{yz}(\mathbf{Q}_s \notin \underline{\mathcal{B}}_{s,t})\mathbb{P}_{yz}(\mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}) \\
&= (1 - \mathbb{P}_{yz}(\mathbf{Q}_s \notin \underline{\mathcal{B}}_{s,t}))\mathbb{P}_{yz}(\mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}) = \mathbb{P}_{yz}(\mathbf{Q}_s \in \underline{\mathcal{B}}_{s,t})\mathbb{P}_{yz}(\mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s}).
\end{aligned}$$

The refined lower bound for (ii) is established similarly. The old lower and upper bounds are tight because the distributions that generate them are consistent with positive respectively negative quadrant dependence. The bounds at $\mathbb{P}_{yz}(\mathbf{Q}_s \in \underline{\mathcal{B}}_{s,t})\mathbb{P}_{yz}(\mathbf{Q}_t \in \underline{\mathcal{B}}_{t,s})$ are tight because independence of $\mathbf{Q}_s$ and $\mathbf{Q}_t$ cannot be excluded.

Example 3 The first and third claim are easy to see, we will establish the one regarding association. To see the lower bound, let $U \equiv \{\mathbf{q} : (1, 2, 1) \cdot \mathbf{q} \geq 5.1\}$, then U contains $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}_2$

but not $\mathbf{a}_2$ or $\mathbf{b}_1$. Now write

$$\begin{aligned}
\mathbb{P}_{yz}(\mathbf{Q}_1 \in U, \mathbf{Q}_2 \in U) &\geq \mathbb{P}_{yz}(\mathbf{Q}_1 \in U)\mathbb{P}_{yz}\left(\mathbf{Q}_2 \in U\right) \\
\Longleftrightarrow \mathbb{P}_{yz}(\mathbf{Q}_2 \in U | \mathbf{Q}_1 \in U) &\geq \mathbb{P}_{yz}\left(\mathbf{Q}_2 \in U\right) \\
\Longleftrightarrow \mathbb{P}_{yz}(\mathbf{Q}_2 \notin U | \mathbf{Q}_1 \in U) &\leq \mathbb{P}_{yz}\left(\mathbf{Q}_2 \notin U\right) \\
\Longleftrightarrow \mathbb{P}_{yz}(\mathbf{Q}_2 \notin U | \mathbf{Q}_1 \notin U) &\geq \mathbb{P}_{yz}\left(\mathbf{Q}_2 \notin U\right) \\
\Longleftrightarrow \mathbb{P}_{yz}(\mathbf{Q}_2 = \mathbf{b}_1 | \mathbf{Q}_1 = \mathbf{a}_2) &\geq \mathbb{P}_{yz}\left(\mathbf{Q}_2 = \mathbf{b}_1\right) = 1/2,
\end{aligned}$$

implying the claim. The bound is tight because association allows for independence.

Lemma 3.1 Throughout this proof, we denote by $\mathbb{Y} = (Y_1..., Y_n)$, $\mathbb{Z} = (Z_1..., Z_n)$ and y, z denote a fixed position. Following standard arguments for local polynomials (e.g., Fan and Gijbels (1996)), we obtain for the bias

$$\mathbb{E}\left[\hat{\pi}_{yz} - \pi_n | \mathbb{Y}, \mathbb{Z}\right] = h^2 n^{-\beta} \frac{\kappa_2}{2} \iota' H\left(c_{yz}\right) \iota + o_p(h^2).$$

The variance requires a bit more care. We decompose the estimator into bias and variance part, i.e.: $\hat{\pi}(y,z) - \pi(y,z) = \sum_i W_{in}(\eta_i + bias_i)$, where W_{in} are weights, see Fan and Gijbels (1999). Next, consider

$$\begin{aligned}
Var\left[\hat{\pi}_{yz}(y,z) | \mathbb{Y}, \mathbb{Z}\right] &= \mathbb{E}\left[\left(\hat{\pi}_{yz} - \pi_n\right)^2 | \mathbb{Y}, \mathbb{Z}\right] \\
&= \mathbb{E}\left[\sum_j \sum_i W_{in} W_{jn} \eta_i \eta_j | \mathbb{Y}, \mathbb{Z}\right] \\
&\quad + 2\mathbb{E}\left[\sum_j \sum_i W_{in} W_{jn} \eta_i bias_j | \mathbb{Y}, \mathbb{Z}\right] \\
&\quad + \mathbb{E}\left[\sum_j \sum_i W_{in} W_{jn} bias_i bias_j | \mathbb{Y}, \mathbb{Z}\right] \\
&= T_1 + T_2 + T_3
\end{aligned}$$

Observe that $T_2 = 0$ by iterated expectations, and $T_3 = o_p(T_1)$. Finally,

$$\mathbb{E}\left[\sum_j \sum_i W_{in} W_{jn} \eta_i \eta_j | \mathbb{Y}, \mathbb{Z}\right] = \sum_i W_{in}^2 \mathbb{E}\left(\eta_i^2 | Y_i = y, Z_i = z\right) = n^{-\beta} \sum_i W_{in}^2 c_{y_i z_i}.$$

Then, by standard arguments, $\sum_i W_{in}^2 c_{y_i z_i} = n^{-1} h^{-d} \kappa_2 c_{yz} + o_p((nh)^{-1})$, and the statement follows by a CLT for triangular arrays, see again Fan and Gijbels (1996).

To see that $\widehat{\pi}_{yz}/\pi_n \xrightarrow{p} 1$ (and hence $\widehat{\sigma}_{yz}/\sigma_n \xrightarrow{p} 1$), observe that

$$\frac{\widehat{\pi}_{yz} - \pi_n}{\pi_n} = \frac{\sqrt{nh^d n^\beta}}{\pi_n \sqrt{nh^d n^\beta}} \left(\widehat{\pi}_n - \pi_n\right) = \frac{1}{c_{yz}\sqrt{nh^d n^{-\beta}}} \underbrace{\sqrt{nh^d n^\beta} \left(\widehat{\pi}_{yz} - \pi_n\right)}_{\equiv A},$$

where the second step follows by substituting for $\pi_n = c_{yz} n^{-\beta}$. By the lemma's main claim, A is stochastically bounded, thus $n^{1-\beta} h^d \to \infty$ implies $\left(\widehat{\pi}_{yz} - \pi_n\right)/\pi_n \xrightarrow{p} 0$ and hence the claim.

Lemma 3.2 We establish the uniform result by showing a pointwise one but in moving parameters (π_n, ψ_n), implying the uniform result because the pointwise finding can be applied to a least favorable sequence. Also, we will make a finite number of case distinction depending on whether parameters are "large" or "small" in senses that will be defined. Every sequence can be partitioned into finitely many subsequences s.t. each subsequence conforms to one case below.

We first establish a number of lemmas showing that the different versions of $CI_{1-\alpha}(\Theta)$ are valid under different sets of conditions.

Lemma A.1. Assume that $\min\left\{\pi_n, 1-\psi_n\right\}/c_n \to 0$. Then $\lim_{n\to\infty} \mathbb{P}(\Theta_0 \subseteq CI^1_{1-\alpha}(\Theta)) = 1$.

Proof.

$$\Theta_0 = [\max\{\pi_n - \psi_n, 0\}, \min\{\pi_n, 1-\psi_n\}] \subseteq [0, \min\{\pi_n, 1-\psi_n\}] \subseteq [0, c_n] \subset [0, CI^1_{1-\alpha}(\Theta)].$$

Lemma A.2. Assume that $\left(1-\pi_n\right)/2b_n \to 0$ and $\psi_n/2b_n \to 0$. Then $\lim_{n\to\infty} \mathbb{P}(\Theta_0 \subseteq CI^2_{1-\alpha}(\Theta)) = 1$.

Proof. Noting that $\pi_n - \psi_n = 1 - (1-\pi_n) - \psi_n \geq 1 - b_n$ for n large enough, write

$$\Theta_0 = [\max\{\pi_n - \psi_n, 0\}, \min\{\pi_n, 1-\psi_n\}] \subseteq [1-b_n, 1] \subset CI^2_{1-\alpha}(\Theta).$$

Lemma A.3. Assume that $\phi_n \sigma_\pi \left(\hat{\pi} - \pi_n\right) \xrightarrow{d} \mathcal{N}(0,1)$, that $\phi_n \sigma_\pi \left(\widehat{\psi} - \psi_n\right) \xrightarrow{p} 0$, and that $\widehat{\sigma}_\pi/\sigma_\pi \to 1$. Then $\lim_{n\to\infty} \mathbb{P}(\Theta_0 \subseteq CI^3_{1-\alpha}(\Theta)) = 1-\alpha$.

Proof. Write

$$\begin{aligned}
& \lim_{n\to\infty} \mathbb{P}(\Theta_0 \subseteq CI^3_{1-\alpha}(\Theta)) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\widehat{\pi} - \widehat{\psi} - c_{1-\alpha}\widehat{\sigma}_\pi \phi_n^{-1} \leq \pi_n - \psi_n, \pi_n \leq \widehat{\pi}_n + c_{1-\alpha}\widehat{\sigma}_\pi \phi_n^{-1}\right) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\phi_n \widehat{\sigma}_\pi \left(\widehat{\pi} - \pi_n - \left(\widehat{\psi} - \psi_n\right)\right) \leq c_{1-\alpha}, \phi_n \widehat{\sigma}_\pi \left(\widehat{\pi}_n - \pi_n\right) \geq -c_{1-\alpha}\right) \\
= & \lim_{n\to\infty} \mathbb{P}\left(-c_{1-\alpha} \leq \phi_n \widehat{\sigma}_\pi \left(\widehat{\pi} - \pi_n\right) \leq c_{1-\alpha}\right) \\
= & \; 1 - \alpha.
\end{aligned}$$

where the last step uses the definition of $c_{1-\alpha}$ and this lemma's assumptions.

Lemma A.4. Assume that $\phi_n \sigma_\psi \left(\widehat{\psi} - \psi_n\right) \xrightarrow{d} \mathcal{N}(0,1)$, that $\phi_n \sigma_\psi \left(\widehat{\pi} - \pi_n\right) \xrightarrow{p} 0$, and that $\widehat{\sigma}_\psi / \sigma_\psi \to 1$. Then $\lim_{n\to\infty} \mathbb{P}(\Theta_0 \subseteq CI^4_{1-\alpha}(\Theta)) = 1 - \alpha$.

Proof. This mimics lemma A.4.

Lemma A.5. Assume that $\left[\phi_n \sigma_\pi \left(\hat{\pi} - \pi_n\right), \phi_n \sigma_\psi \left(\widehat{\psi} - \psi_n\right)\right] \xrightarrow{d} \mathcal{N}(0, I_2)$, that $\widehat{\sigma}_\pi / \sigma_\pi \to 1$, and that $\widehat{\sigma}_\psi / \sigma_\psi \to 1$. Then $\lim_{n\to\infty} \mathbb{P}(\Theta_0 \subseteq CI_{1-\alpha}(\Theta)) = 1 - \alpha$.

Proof. This case requires two sub-distinctions amounting to four distinct sub-cases, according to which of (**??**-**??**) must be presumed to (almost) bind. Let ε_n be a sequence s.t. $\varepsilon_n \to 0$, $\varepsilon_n / \phi_n \to \infty$, but $\varepsilon_n / a_n \to 0$. First, assume that $\pi_n - \psi_n \geq -\varepsilon_n$, meaning that (**??**) must be taken into account. Then $CI_{1-\alpha}(\Theta)$ will be constructed according to (**??**-**??**) with probability approaching 1, thus it suffices to show validity of this construction. Assume first that $|\pi + \psi - 1| \leq \varepsilon_n$, thus $\left|\widehat{\pi} + \widehat{\psi} - 1\right| \leq \varepsilon_n$ with probability approaching 1. We can then write

$$\lim_{n\to\infty} \mathbb{P}(\Theta_0 \subseteq CI_{1-\alpha}(\Theta)) = \lim_{n\to\infty} \mathbb{P}\begin{pmatrix} \max\left\{\widehat{\pi} - \widehat{\psi} - \phi_n^{-1} c^l, 0\right\} \leq \max\{\pi_n - \psi_n, 0\}, \\ \min\{\pi_n, 1 - \psi_n\} \leq \min\left\{\widehat{\pi}, 1 - \widehat{\psi}\right\} + \phi_n^{-1} c^u \end{pmatrix}.$$

We bound the r.h. probability from below by observing some logical implications. First,

$$\widehat{\pi} - \widehat{\psi} - \phi_n^{-1} c^l \leq \pi_n - \psi_n \Longrightarrow \max\left\{\widehat{\pi} - \widehat{\psi} - \phi_n^{-1} c^l, 0\right\} \leq \max\{\pi_n - \psi_n, 0\}.$$

To see this, note that if $\pi_n - \psi_n \geq 0$, then the two inequalities are equivalent except if $\widehat{\pi} - \widehat{\psi} - \phi_n^{-1} c^l \leq 0$, in which case they are both fulfilled. If $\pi_n - \psi_n < 0$, then the l.h. inequality implies the r.h. one because whenever the l.h. inequality holds, both sides of the r.h. inequality equal 0.

Second,

$$
\begin{aligned}
& \min\left\{\widehat{\pi}, 1-\widehat{\psi}\right\} - \min\{\pi_n, 1-\psi_n\} \\
= & \min\left\{\widehat{\pi} - \min\{\pi_n, 1-\psi_n\}, 1-\widehat{\psi} - \min\{\pi_n, 1-\psi_n\}\right\} \\
\geq & \min\left\{\widehat{\pi} - \pi_n, \psi_n - \widehat{\psi}\right\}.
\end{aligned}
$$

Together, these implications yield

$$
\begin{aligned}
& \lim_{n\to\infty} \mathbb{P}(\Theta_0 \subseteq CI_{1-\alpha}(\Theta)) \\
\geq & \lim_{n\to\infty} \mathbb{P}\left(\widehat{\pi} - \widehat{\psi} - \phi_n^{-1}c^l \leq \pi_n - \psi_n, \min\left\{\widehat{\pi} - \pi_n, \psi_n - \widehat{\psi}\right\} \geq -\phi_n^{-1}c^u\right) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\widehat{\pi} - \widehat{\psi} - \phi_n^{-1}c^l \leq \pi_n - \psi_n, \widehat{\pi} - \pi_n \geq -\phi_n^{-1}c^u, \widehat{\psi} - \psi_n \leq \phi_n^{-1}c^u\right) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\phi_n\left(\widehat{\pi} - \pi_n - \left(\widehat{\psi} - \psi_n\right)\right) \leq c^l, \phi_n\left(\widehat{\pi} - \pi_n\right) \geq -c^u, \phi_n\left(\widehat{\psi} - \psi_n\right) \leq c^u\right) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\sigma_\pi Z_1 - \sigma_\psi Z_2 \leq c^l, \sigma_\pi Z_1 \geq -c^u, \sigma_\psi Z_2 \leq c^u\right) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\widehat{\sigma}_\pi Z_1 - \widehat{\sigma}_\psi Z_2 \leq c^l, \widehat{\sigma}_\pi Z_1 \geq -c^u, \widehat{\sigma}_\psi Z_2 \leq c^u\right) \\
\geq & 1-\alpha,
\end{aligned}
$$

where the last steps use this lemma's assumptions and condition (**??**).

Now, let $\pi_n - \psi_n < -\varepsilon_n$. In this case, $CI_{1-\alpha}(\Theta)$ will be constructed according to (**??**-**??**) with some probability and according to (**??**-**??**) with the remaining probability (which goes to 1 as $\pi_n - \psi_n$ becomes very small). In any case, construction (**??**-**??**) is by construction larger than (**??**-**??**), thus it suffices to show the claim under the premise that construction (**??**-**??**) applies with probability 1. The argument is similar to the above.

Proof of main result. Every sequence (π_n, ψ_n) can be decomposed into subsequences s.t. one of the above lemmas applies to each subsequence. The pre-tests are designed to use the appropriate procedure depending on features of (π_n, ψ_n). If (π_n, ψ_n) is far away from the benchmark sequences specified in the pre-tests, this match will be perfect and one of the above lemmas will apply directly. If $CI_{1-\alpha}(\Theta)$ might oscillate between different procedures in the limit, some additional argument is needed. To keep track of the 49 potential case distinctions, categorize the possible subsequences as in the following table.

		$\mathbf{1}-\boldsymbol{\psi}_n=\ldots$			$\boldsymbol{\psi}_n=\ldots$			
		$\mathbf{o}(\mathbf{c}_n)$	$\mathbf{O}(\mathbf{c}_n)$	**(other)**	$\mathbf{O}(\mathbf{b}_n)$	$\mathbf{o}(\mathbf{b}_n)$	$\mathbf{O}(\mathbf{c}_n)$	$\mathbf{o}(\mathbf{c}_n)$
$\boldsymbol{\pi}_n=\ldots$	$\mathbf{o}(\mathbf{c}_n)$	2	2	2	2	2	2	2
	$\mathbf{O}(\mathbf{c}_n)$	2	4	4	4	4	10	8
	(other)	2	4	1	1	1	11	6
$\mathbf{1}-\boldsymbol{\pi}_n=\ldots$	$\mathbf{O}(\mathbf{b}_n)$	2	4	1	5	5	12	6
	$\mathbf{o}(\mathbf{b}_n)$	2	4	1	5	3	3	3
	$\mathbf{O}(\mathbf{c}_n)$	2	13	14	15	3	3	3
	$\mathbf{o}(\mathbf{c}_n)$	2	9	7	7	3	3	3

In this table, "other" refers to all sequences s.t. $\psi_n > O(b_n)$ and $1-\psi_n > O(c_n)$ (respectively the same for π_n). In cases labelled 1, the baseline construction is valid and will be used with probability approaching 1. The same is true for $CI^1_{1-\alpha}(\Theta)$ in the cases labelled 2 and for $CI^2_{1-\alpha}(\Theta)$ in the cases labelled 3. In cases labelled 4, $CI_{1-\alpha}(\Theta)$ may oscillate between constructions $CI^1_{1-\alpha}(\Theta)$ and $CI^5_{1-\alpha}(\Theta)$. Note, though, that in these cases one will have $CI^5_{1-\alpha}(\Theta) \subseteq CI^1_{1-\alpha}(\Theta)$ by construction and furthermore that lemma A.5 applies, thus $CI_{1-\alpha}(\Theta)$ is valid (if potentially conservative). In case 5, an analogous argument applies but with $CI^2_{1-\alpha}(\Theta)$ and $CI^5_{1-\alpha}(\Theta)$. In case 6, one can directly apply lemma A.3, and in case 7, the same holds for lemma A.4. In case 8, $CI_{1-\alpha}(\Theta)$ may oscillate between $CI^1_{1-\alpha}(\Theta)$ and $CI^3_{1-\alpha}(\Theta)$, but $CI^3_{1-\alpha}(\Theta) \subseteq CI^1_{1-\alpha}(\Theta)$ by construction and lemma A.3 applies. A similar argument applies to case 9. In all of cases 10-12, $CI^3_{1-\alpha}(\Theta)$ and $CI^5_{1-\alpha}(\Theta)$ are asymptotically equivalent. Validity in case 11, where $CI_{1-\alpha}(\Theta) \in \left\{CI^3_{1-\alpha}(\Theta), CI^5_{1-\alpha}(\Theta)\right\}$ with probability approaching 1, follows from lemma A.5. In cases 10 and 12, where the probability of $CI_{1-\alpha}(\Theta) = CI^1_{1-\alpha}(\Theta)$ (case 10) or $CI_{1-\alpha}(\Theta) = CI^2_{1-\alpha}(\Theta)$ (case 12) fails to vanish, additional argument along previous lines is needed. The analog argument holds for cases 13-15.

Consider now the claim that $\lim_{n\to\infty} \inf_{\theta_{yz}\in\Theta_{yz}} \mathbb{P}(\theta_{yz} \in CI_{1-\alpha}(\theta)) = 1-\alpha$. The proof plan for this is similar to the above, and we only elaborate steps that differ. In particular, lemmas A.1 and A.2 immediately imply the analogous result here. It remains to demonstrate the following.

Lemma B.3. Assume that $\phi_n\sigma_\pi(\hat{\pi}-\pi_n) \xrightarrow{d} \mathcal{N}(0,1)$, that $\phi_n\sigma_\pi\left(\widehat{\psi}-\psi_n\right) \xrightarrow{p} 0$, and that $\widehat{\sigma}_\pi/\sigma_\pi \to 1$. Then $\lim_{n\to\infty} \inf_{\theta_{yz}\in\Theta_{yz}} \mathbb{P}(\theta_{yz} \in CI^3_{1-\alpha}(\theta)) = 1-\alpha$.

Proof. Again, with probability approaching 1 we have $\Theta_0 = [\pi_n - \psi_n, \pi_n]$, $\Delta = \psi_n$, $\widehat{\Theta} = \left[\widehat{\pi}-\widehat{\psi}, \widehat{\pi}\right]$, and $\widehat{\Delta} = \min\left\{\widehat{\pi}, 1-\widehat{\pi}, \widehat{\psi}, 1-\widehat{\psi}\right\} = \widehat{\psi}$, thus $\widehat{\Delta}$ is superefficient relative to the rate of convergence of $\hat{\pi}$. Parameterizing the true parameter value as $\theta = \pi_n - a\psi_n$ for $a \in [0,1]$,[1] one

[1]Strictly speaking we should allow a to be a moving parameter as well, but obviously any sequence $\{a_n\}$ will have finitely many accumulation points in $[0,1]$ and the argument can be conducted separately along the according

can then write

$$
\begin{aligned}
& \lim_{n\to\infty} \mathbb{P}(\theta \in CI^3_{1-\alpha}(\theta)) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\widehat{\pi} - \widehat{\psi} - c_{1-\alpha}\widehat{\sigma}_\pi {\phi_n}^{-1} \le \pi_n - a\psi_n \le \widehat{\pi} + c_{1-\alpha}\widehat{\sigma}_\pi {\phi_n}^{-1}\right) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\phi_n \widehat{\sigma}_\pi^{-1}\left(a\psi_n - \widehat{\psi}\right) - c_{1-\alpha} \le \phi_n \widehat{\sigma}_\pi^{-1}\left(\pi_n - \widehat{\pi}\right) \le c_{1-\alpha} + \phi_n \widehat{\sigma}_\pi^{-1} a\psi_n\right) \\
= & \lim_{n\to\infty} \mathbb{P}\left(\phi_n \sigma_\pi^{-1}\left(a-1\right)\psi_n - c_{1-\alpha} \le \phi_n \sigma_\pi^{-1}\left(\pi_n - \widehat{\pi}\right) \le c_{1-\alpha} + \phi_n \sigma_\pi^{-1} a\psi_n\right) \\
= & \lim_{n\to\infty} \left(\Phi\left(c_{1-\alpha} + \phi_n \sigma_\pi^{-1} a\psi_n\right) - \Phi\left(\phi_n \sigma_\pi^{-1}\left(a-1\right)\psi_n - c_{1-\alpha}\right)\right).
\end{aligned}
$$

Direct evaluation of derivatives shows that this limit is concave in a and is minimized when $a \in \{0,1\}$, in which case it equals $1-\alpha$.

Lemma B.4. Assume that $\phi_n \sigma_\psi \left(\widehat{\psi} - \psi_n\right) \xrightarrow{d} \mathcal{N}(0,1)$, that $\phi_n \sigma_\psi \left(\widehat{\pi} - \pi_n\right) \xrightarrow{p} 0$, and that $\widehat{\sigma}_\psi / \sigma_\psi \to 1$. Then $\lim_{n\to\infty} \inf_{\theta_{yz}\in\Theta_{yz}} \mathbb{P}(\theta \in CI^4_{1-\alpha}(\theta)) = 1-\alpha$.

Proof. This mimics lemma B.3.

Lemma B.5. Let the assumptions of lemma A.5 hold. Then $\mathbb{P}(\theta \in CI^5_{1-\alpha}(\theta)) = 1-\alpha$.

Proof. In view of the fact that if $\Delta \to 0$, then $\phi_n \left(\widehat{\Delta} - \Delta\right) \to 0$, this follows by minimal adaptation of arguments in Stoye (2009, proposition 1).

Proof of main result. This is now analogous to the proof of the main result in the preceding proof.

subsequences.