Choice Theory when Agents can Randomize

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Abstract

This paper takes choice theory to risk or uncertainty. Well-known decision models are axiomatized under the premise that agents can randomize. Under a reversal of order assumption, this convexifies choice sets, and even after imposing the weak axiom of revealed preference and nonemptiness of choice correspondences, the preferences directly revealed by choice may be incomplete or cyclical.

Choice correspondence characterizations of (von Neumann-Morgenstern) expected utility, (Anscombe-Aumann) subjective expected utility, (Gilboa-Schmeidler) maxmin expected utility, (Schmeidler) Choquet expected utility, (Wald-Milnor) maximin utility, and (Bewley) multiple prior preferences can be established nonetheless by assuming nonempty choice, the weak axiom of revealed preference or weakenings thereof (but avoiding the strong axiom throughout), and choice correspondence analogs of relevant preference axioms. Two salient applications are to games, where agents' ability to randomize is usually presumed, and to statistical decision theory, where agents (i.e., statisticians) randomize in reality.

Keywords: revealed preference, choice functions, choice correspondences, completeness, transitivity, expected utility, maxmin expected utility, minimax.

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1 Introduction

1.1 Overview

This paper axiomatizes the choice correspondences generated by expected utility, Choquet expected utility, maxmin expected utility, maxmin utility, and multiple prior expected utility, when all choice problems are closed under probabilistic mixture. This provides a choice theoretic foundation for these models in settings where agents can randomize. In such settings, only choice from convex (in the sense of closed under mixture) sets is observable, and axioms will be asserted only for such sets. Furthermore, the weak axiom of revealed preference (WARP) will be assumed but does not any more imply the strong axiom of revealed preference (SARP), nor will the latter be imposed.

The analysis is motivated by a simple but apparently novel observation: Randomized choice from (the convex hull of) $\{f, g\}$ need not reveal preference between f and g because the agent might choose a mixture of the two. Indeed, even if WARP holds, preferences revealed by choice from (convex hulls of) binary menus need not be complete, transitive, or acyclical. As a result, classic characterizations due to von Neumann and Morgenstern (1947), Anscombe and Aumann (1963), Schmeidler (1989), Gilboa and Schmeidler (1989), Bewley (2002), and Milnor (1954), do not apply to this relation. The question is whether this can be fixed: Do choice-based analogs of said characterizations go through, albeit with additional argument? The answer is "yes" if one transcribes axioms on preferences into axioms on choice correspondences, adds WARP, and assumes that choice correspondences are nonempty.

Here is a brief exposition of the problem observed, and the solution offered, in a specific example. Let Wayne's world be characterized by two states and assume that acts are sufficiently described by expected utility vectors $(u, v) \in \mathbb{R}^2$. Wayne chooses $\arg\max_{(u,v)\in M} \min\{u, v\}$ from every menu M. Attempting to describe his behavior, one would be tempted to write $(2, 1) \succ (0, 2)$, but no choice experiment will directly reveal this preference because menus are convex, hence whenever both acts are available, Wayne will pick $(4/3, 4/3)$ (by picking $(2, 1)$ with probability $2/3$) or something altogether different. More generally, even though Wayne's choice correspondence is well-behaved and in particular fulfils WARP, his directly revealed preferences are incomplete, intransitive, and fail standard Archimedean (continuity) axioms. This illustrates two points: (i) WARP and other baseline properties of choice correspondences on convex sets do not imply completeness, transitivity, nor continuity of revealed preferences. (They don't even imply acyclicity, though it would take more exotic agents than Wayne to make the point.) (ii) Just assuming completeness etc. of revealed preferences is not an option even for recovering a relatively standard model like maxmin expected utility.

The resolution is as follows: A choice correspondence analog of Gilboa and Schmeidler's (1989) c-independence axiom, together with a weakened Archimedean property, restores a representation of choice as maximizing a well-behaved "as if"-preference ordering. If one furthermore imposes analogs of their other axioms, then this “as if”-ordering is maxmin expected utility, thus a characterization of their model has been recovered. The analogous problem will be solved for numerous well-known models, but remains open for others.

The findings also imply novel axiomatizations of these models in preference terms. They then imply that in the aforementioned characterizations, completeness and transitivity are implied by certain other axioms. Under a normative interpretation, this means that transitivity need not be justified if certain other axioms are imposed. From the descriptive point of view, it means that care should be taken when attempting to test transitivity separately from other conditions in decision making under risk or uncertainty. Whatever the interpretation, the result may appear surprising because a comparable finding is *not* available for the theory of demand, where convexity of budget sets causes a similar issue. The difference between the settings lies in a richer choice domain but more importantly, in the availability of axioms that powerfully interact with nonemptiness of choice correspondences and with WARP.

The next subsection of this paper adds to the motivation, section 2 contains the axiomatic development, and section 3 concludes. The appendix collects all proofs. Examples for independence of axioms are available from the author.

1.2 Why Consider Randomization?

The extension of choice theory to settings where agents can randomize is relevant for numerous reasons. First, decision theoretic models are routinely “plugged into” economic models that allow agents to randomize, creating a wedge between axiomatizations and their applications. For example, game theory builds on von Neumann-Morgenstern utility, yet randomization is generally needed for Nash equilibrium to exist. A characterization of von Neumann-Morgenstern utility that allows for randomization should, therefore, be of obvious interest.

Axiomatizations can also be used to normatively justify decision rules. The obvious instance, and original motivation of Savage’s (1954) work, is statistical decision theory. Convex sets are equally salient here because statisticians can, do, and should (according to normative criteria they routinely use) randomize. For a simple example, hypothesis tests about binomial parameters that maximize power given size generically randomize with positive probability. More generally, randomization is a common feature of non-Bayesian statistical decision rules. If interest is in minimax or Γ-minimax, both of which are axiomatized in this paper, then to preclude randomization is to remove from axiomatization the very acts that will ultimately be chosen.[1]

[1] See Hirano (2008) for an expositon of statistical decision theory notation to economists, Berger (1985) for a definition of Γ-minimax (which corresponds to Gilboa and Schmediler’s (1989) maxmin expected utility), and Stoye (2012a) for an extended survey, including several examples of randomized optimal decision rules.

Finally, axiomatic characterizations can serve as precise delineation of the empirical content of theories to inform their testing, e.g. in laboratory experiments. Here, randomization is a concern if subjects are able to randomize and if a preference for doing so is not precluded by assumption. Regarding the former, whether or not one accepts the possibility of mental randomization, incentivization schemes may introduce objective probabilities, as well as at least some coarse randomization devices, into the laboratory environment.[2] Regarding the latter, assuming that subjects dislike randomization is not an option if one wants to test or estimate models that imply the opposite preference, notably models that feature uncertainty aversion as defined by Schmeidler (1989). Regarding the empirical record, one "smoking gun" of randomization, namely choice switching over repeated instances of the same problem, regularly occurs in the lab.[3] While there are many potential explanations for this phenomenon, it might be interesting to not exclude randomization a priori. Indeed, Agranov and Ortoleva (2013) find that an explicit option to randomize is frequently exercised, suggesting that "stochastic hedging" is empirically relevant (though, see Dominiak and Schnedler (2011)).

Even if one accepts the relevance of randomization, another question is how to model it. The bulk of this paper assumes "reversal of order," i.e. that the decision maker compounds her own randomization with other objective risk. This assumption is certainly strong, especially when combined with ambiguity aversion. Continuing the example of Wayne above (and taking a cue from a referee), it is one thing to say that Wayne prefers $(4/3, 4/3)$ to either of $\{(1, 2), (2, 0)\}$, but it is a much stronger claim that he identifies his own randomization of $\frac{2}{3} \otimes (1, 2) \oplus \frac{1}{3} \otimes (2, 0)$ (where $\oplus$ and $\otimes$ refers to ex-ante mixture by the decision maker) with that same act. In addition, he must be able to commit to this randomization, which may be an issue in situations where his explicit choice is between $(1, 2)$ and $(2, 0)$ and he plausibly prefers the former pure act. Readers who find these assumptions unbelievable may find that corollary 2 reported below gives them a reason to ignore randomization.

However, the present paper is not the only one to be informed by such assumptions – see Raiffa's (1961) critique of Ellsberg (1961) for a classic example and Agranov and Ortoleva (2013) or Kucmiz (2013) for current ones. Also, the same assumptions are implicit in statistical decision theory since Wald (1950), whose very notation implies that randomization by the statistician is compounded with sampling risk. Finally, the same assumptions are reflected in game theoretic notation, though method-

[2] An obvious example is random lottery selection when choice problems are repeated. Also, Caplin and Dean (2011) propose and Caplin, Dean, and Martin (2011) implement a test of search models by recording sequences of choice from a fixed menu. Subjects are incentivized by executing the choice that was current at a randomly selected time. While preference for randomization is not a plausible concern in their specific setup, one might note for future reference that the design provides subjects with an explicit, continuous, objective randomization device.

[3] See the survey by Wilcox (2008) for an extensive discussion and Ballinger and Wilcox (1997), Birnbaum and Schmidt (2008, 2010), Camerer (1989), Loomes and Sugden (1998), or Starmer and Sugden (1989), for examples of choice switching as problems are repeated within a short time frame.

ological qualms about the role of randomization in games are about as old as the notion of mixed strategies.

In sum, I acknowledge that one can reasonably disagree with this paper's modelling choices, just as not everybody agrees with Raiffa (1961). The present paper's assumptions are certainly not innocuous, but I believe that their implications are worth exploring.[4]

2 Axiomatic Analysis

2.1 Setting the Stage

Consider a set $\mathcal{X}$ of outcomes x and the set $\Delta\mathcal{X}$ of simple lotteries $p, q, r, \ldots$ over such outcomes. (The symbol $\Delta(.)$ will generally denote the set of finite mixtures over elements of the argument.) In section 2.2, these lotteries will be the objects of choice. More generally, there is a state space $\mathcal{S}$ with typical element s and endowed with sigma-algebra Σ, and acts $f, g, h, \ldots \in \mathcal{F}$ are simple, Σ-measurable functions from $\mathcal{S}$ into $\Delta\mathcal{X}$ that map states s onto lotteries $f(s) \in \Delta\mathcal{X}$. A lottery-act is *constant* if $f(s)$ does not depend on s. With the usual abuse of notation, $\Delta\mathcal{X}$ will be embedded in $\mathcal{F}$ by identifying the constant acts with the corresponding lotteries. Mixtures between acts are defined as statewise probabilistic mixtures: $\lambda f + (1-\lambda)g$ is characterized by $(\lambda f + (1-\lambda)g)(s) = \lambda f(s) + (1-\lambda)g(s)$. Henceforth, "convex" means "closed under mixture." A choice problem is a compact (with respect to uniform weak convergence), convex menu M. The notation $\lambda M + (1-\lambda)f$ will denote the menu generated from M by mixing all of its elements with f: $\lambda M + (1-\lambda)f = \{\lambda g + (1-\lambda)f : g \in M\}$.

As convexity of menus is crucial, note the following: (i) Probabilistic mixtures of acts and randomizations over them are considered the same, thus the symbol $\Delta\mathcal{X}$ equivalently denotes the convex hull of $\mathcal{X}$. This reflects the previously discussed reversal of order assumption. (ii) The choice problem explicitly presented to the decision maker, say in a laboratory setting or a finite game, may not be convex, but if the decision maker can randomize, then she is effectively choosing from its convex hull. (iii) I assume that choice of lotteries is observable. In some settings, one would realistically observe realizations of these lotteries, creating an additional layer of complication.

To model agents' behavior, one could take as primitive either a preference $\succsim$ over acts or a choice correspondence C that maps menus M into choice sets $C(M) \subseteq M$. If choice from binary menus were observable, these primitives would be interchangeable in the following sense: For any binary relation $\succsim$, define the choice correspondence $C_{\succsim}$ by $C_{\succsim} = \{f \in M : g \in M \Rightarrow f \succsim g\}$. Similarly, for any choice correspondence C, define the binary relation $\succsim_C$ by $f \succsim_C g$ if $f \in C(\{f, g\})$. Now, if a

[4] For the implications of other modelling choices, see corollary 2 below but in particular a current, interesting paper by Saito (2013) that I discovered while revising this paper. Also, Segal (1990) and Seo (2009) relax reversal of order in related settings.

binary relation is complete and transitive, then $C_{\succsim}$ satisfies WARP and nonemptiness for finite sets and $\succsim = \succsim_{C_{\succsim}}$. Similarly, if a choice correspondence C satisfies nonemptiness and WARP, then $\succsim_C$ is complete and transitive and $C = C_{\succsim_C}$ (Arrow (1959), Sen (1971)). Much classic decision theory fits into this framework, although a recent and growing literature analyzes choice in settings where things are not so easy (e.g., Caplin and Dean (2011), Eliaz and Ok (2006), Hayashi (2008), Stoye (2011b)).

This paper will choose C as primitive and also work with the directly revealed preference relation induced by C:

Definition 1 ***Revealed Preference Relation***

$f \succsim_C g$ if $f \in C(\Delta\{f,g\})$; $f \succ_C g$ if $f \succsim_C g$ and not $g \succsim_C f$; $f \sim_C g$ if $f \succsim_C g$ and $g \succsim_C f$.

Most of this paper will use WARP, here identified with Arrows's (1959, axiom C4) "Independence of Irrelevant Alternatives" axiom, as well as nonemptiness of C.

Axiom 1 ***Weak Axiom of Revealed Preference***

$C(M \cup N) \cap M \in \{C(M), \varnothing\}$.

Axiom 2 ***Nonemptiness***

$C(M)$ is nonempty-valued.

These assumptions suffice to recover a familiar dualism between choice and preference.

Remark 1 *If C fulfils WARP and nonemptiness, then it is rationalized by $\succsim_C$: $C(M) = \{f : g \in M \Rightarrow f \succsim_C g\}$.*

Proof. See Richter (1971, theorem 2). ■

However, the aforementioned interchangeability fails because $\succsim_C$ need not be complete: If choice from $\Delta\{f,g\}$ is a proper mixture of f and g, then f and g are not $\succsim_C$-comparable. This case will be denoted by $f \bowtie_C g$. Without further restrictions, transitivity (and even acyclicity) of $\succ_C$ and $\succsim_C$ is not implied either, and transitivity (though not acyclicity) does indeed fail in some examples considered later.[5] Thus, $\succsim_C$ may be incomplete or cyclical.[6] The task ahead is to investigate whether imposing choice correspondence analogs of certain preference axioms remedies this.

[5] Consider skew-symmetric bilinear utility (Fishburn (1982)) for an example of a rationalizable choice function that reveals a cyclical preference but is nonetheless nonempty on convex sets (as proved by Fishburn and LaValle (1988)). The author can provide an (admittedly somewhat contrived) example that fulfils WARP.

[6] The latter could be remedied by assumption, i.e. by imposing SARP. However, I agree with Sen (1973) and others that this would come uncomfortably close to assuming preferences and thus be at tension with the motivation of a choice theoretic approach. In any case, one finding of this paper is that imposing SARP will not be necessary.

2.2 Revealed Preference with Independence

This section characterizes von Neumann-Morgenstern utility for choice from convex sets of constant acts. While new, this result is elementary. However, it is of interest due to the prominence of von Neumann-Morgenstern utility and because it will serve as building block for more intricate results later on.

Define C on all compact convex subsets of $\Delta\mathcal{X}$ and consider the following axioms.

Axiom 3 ***Constant Act Independence***

$$C(\lambda M + (1-\lambda)p) = \lambda C(M) + (1-\lambda)p$$

for all compact convex menus $M \subseteq \Delta\mathcal{X}$, lotteries $p \in \Delta\mathcal{X}$, and scalars $\lambda \in (0,1)$.

Axiom 4 ***Constant Act Archimedean Property***

For all lotteries $p, q, r \in \Delta\mathcal{X}$:

(i) *If $\lambda p + (1-\lambda)r \succ_C q$ for all $\lambda \in (0,1]$, then not $q \succ_C r$.*

(ii) *If $\lambda p + (1-\lambda)r \prec_C q$ for all $\lambda \in (0,1]$, then not $q \prec_C r$.*

The same adaptation of independence to choice correspondences is found in Eliaz and Ok (2006), among others. It is not to be confused with c-independence, which will be introduced later. Adaptation of the Archimedean property is delicate. The reader might have anticipated the following, stronger axiom: $p \succ_C q \succ_C r \Rightarrow \exists \lambda, \gamma \in (0,1) : \lambda p + (1-\lambda)r \succ_C q \succ_C \gamma p + (1-\gamma)r$. Axiom 4 is weaker by permitting noncomparability in the conclusion. This is necessary because even if C were induced by a preference ordering $\succsim$, $(\lambda p + (1-\lambda)r \bowtie_C q, \forall \lambda \in (0,1))$ would be excluded by a conventional Archimedean property of $\succsim$ only in conjunction with completeness, so excluding it here would inject an unwanted vestige of completeness.[7]

Independence, continuity, and WARP jointly characterize the von Neumann-Morgenstern choice correspondence.

Theorem 1 *Let C be defined on all compact convex menus $M \subseteq \Delta\mathcal{X}$. C fulfils WARP, nonemptiness, constant act independence, and the constant act Archimedean property iff there exists $U : \mathcal{X} \mapsto \mathbb{R}$, U unique up to positive affine transformation, such that*

$$C(M) = \arg\max_{p \in M} \int U(x)dp.$$

[7] In this subsection's setting, the matter is inconsequential because completeness will be recovered anyway, but a stronger axiom would cause problems later on. Similarly, (ii) is implied by (i) and independence, but independence will be dropped later, so I impose both parts throughout. The Archimedean property resembles Aumann's (1962), who only imposes (i).

Implicit in this result is a "dual theorem" that characterizes expected utility preferences without assuming transitivity and is useful in comparing theorem 1 to existing findings. Thus, consider the following axioms for a potentially incomplete preference relation $\succsim$ on $\Delta\mathcal{X} \times \Delta\mathcal{X}$.

Archimedean Property: If $\lambda p + (1-\lambda)r \succ [\prec]q, \forall \lambda \in (0,1]$, then not $r \prec [\succ]q$.

Independence: $p \succsim q \Leftrightarrow \lambda p + (1-\lambda)r \succsim \lambda q + (1-\lambda)r$, $\forall p, q, r \in \Delta\mathcal{X}, \lambda \in (0,1)$.

Decisiveness: $C_{\succsim}(M) \equiv \{p \in M : q \in M \Rightarrow p \succsim q\} \neq \varnothing$ for all compact convex M.

Property β: $C_{\succsim}$ fulfils property β. i.e. $\{f,g\} \subseteq C_{\succsim}(M) \Longrightarrow C_{\succsim}(M \cup N) \cap \{f,g\} \in \{\{f,g\},\varnothing\}$.

Property α of $C_{\succsim}$ (i.e., $C_{\succsim}(M \cup N) \cap M \subseteq C_{\succsim}(M)$) is not explicitly imposed because it follows from the assumption (implicit in notation) that $\succsim$ is menu-independent.

Corollary 1 *The relation $\succsim$ fulfils the above axioms iff it is von Neumann-Morgenstern utility.*

I now give an intuition for theorem 1 and then discuss some related findings. The basic idea is that independence precludes strict preference for randomization, thus $C(\Delta\{p,q\})$ must contain one of $\{p,q\}$ and $C(\Delta\{p,q,r\})$ must contain one of $\{p,q,r\}$. The former yields completeness of $\succsim_C$. The latter excludes cycles of the form $p \succsim_C q \succsim_C r \succ_C p$; these are consistent with WARP, but only if choice from $\Delta\{p,q,r\}$ is strictly interior. Once $\succsim_C$ is known to be complete and transitive, it can be verified to fulfil Herstein and Milnor's (1953) axioms.[8]

Dekel, Lipman, and Rustichini (2001; see also Dekel, Lipman, Rustichini, and Sarver (2007)) analyze preferences over sets of lotteries. While choice *from* menus is not axiomatized, one of their results could naturally be interpreted as characterizing the indirect utility function corresponding to von Neumann-Morgenstern expected utility. A core assumption of theirs is that convexifying choice sets does not increase their value. This would apply naturally if decision makers can randomize and reversal of order is assumed, so that explicit convexification of choice sets would really leave them unchanged. Thus, while the framework is rather different, Dekel, Lipman, and Rustichini's result shares important features of theorem 1.

Gul and Pesendorfer (2006) characterize a random utility version of the von Neumann-Morgenstern choice correspondence. They impose a stochastic analog of property α and also use the obvious adaptation of independence to choice probabilities. While their argument is in many dimensions much more complex than the present one, the crucial insight underlying theorem 1 is essentially imposed by assumption, namely by postulating that choice correspondences are supported on the extreme points

[8] One might wonder whether this reasoning goes through if one weakens independence to betweenness of $\succsim_C$ (i.e., $p \sim_C [\succ_C]q \Rightarrow p \sim_C [\succ_C]\lambda p + (1-\lambda)q$ for all $\lambda \in [0,1]$; Chew (1983), Dekel (1986)). This conjecture is of interest because betweenness of preferences suffices to ensure existence of Nash equilibrium in games. However, it is false; a counterexample is available from the author.

of the menu. The proof of theorem 1 does not render this assumption redundant in their context because they are unable to impose WARP.

The only other preference characterization of von Neumann-Morgenstern utility that does not use transitivity was (to my knowledge) provided by Fishburn (1982). Fishburn does not impose decisiveness nor property β, but a collection of "convexity" axioms that are jointly stronger than betweenness and (given other axioms) decisiveness but non-nested with independence.[9] This characterizes skew-symmetric bilinear utility; adding independence leads to expected utility without an explicit use of completeness or transitivity. The choice correspondence generated by Fishburn's model fulfils property α but not β. Corollary 1 implies that after imposing independence, imposing property β would allow one to weaken convexity to decisiveness.

Dubra, Maccheroni, and Ok (2004) drop completeness from the von Neumann-Morgenstern axioms (and strengthen continuity). This characterizes the incomplete ordering where $p \succsim q$ if p Pareto-dominates q with respect to a family of utility functions. Dubra, Maccheroni, and Ok (2004) define the choice correspondence to collect non-dominated acts; this correspondence is never empty and fulfils property α but not β. Alternatively, one could define $\trianglerighteq$ as the union of $\succsim$ and $\bowtie$, where $\bowtie$ is the implied noncomparability relation. Then $\trianglerighteq$ is complete and acyclic but intransitive, and it induces the same choice correspondence. This illustrates that incompleteness and intransitivity of preference may be hard to disentangle by observing choice (see also Eliaz and Ok (2006)), as well as necessity of WARP (including property β) in theorem 1. I will consider incomplete "true" preferences, though not multi-utility models, later.

I conclude this section with the following observation, whose proof is a by-product of theorem 1.

Corollary 2 *Let $\mathcal{A}$ denote any set of acts and embed $\mathcal{A}$ in $\Delta\mathcal{A}$ by identifying its elements with degenerate lotteries. Define C on $\{\Delta M : M$ is a finite subset of $\mathcal{A}\}$ and impose WARP, nonemptiness, and independence, but not reversal of order. Then C can be rationalized as maximizing a complete and transitive preference ordering $\succsim_C$ over $\mathcal{A}$. No further property of $\succsim_C$ is implied.*

The suggested interpretation of the corollary is as follows: The researcher wants to learn preferences over elements of $\mathcal{A}$ and can present the decision maker with arbitrary finite choice problems in $\mathcal{A}$; however, the decision maker can randomize, and Δ represents this randomization. Suppose the researcher is willing to assume that the decision maker is neutral with respect to her own randomization, though not necessarily with respect to statewise mixture of acts. The two attitudes can coexist here because reversal of order is not imposed. Reasonable further restrictions on C then ensure recoverability of a preference $\succsim_C$ over $\mathcal{A}$ (through its embedding in $\Delta\mathcal{A}$), resolving the problem which motivates this

[9] The convexity axioms are (quantifiers are $\forall p, q, r \in \Delta\mathcal{X}, \lambda \in (0,1)$): (i) $[p \succ q, p \succsim r] \Rightarrow p \succ \lambda q + (1-\lambda)r$, (ii) $[p \prec q, p \precsim r] \Rightarrow p \prec \lambda q + (1-\lambda)r$, (iii) $[p \sim q, p \sim r] \Rightarrow p \sim \lambda q + (1-\lambda)r$. Note that convexity implies betweenness.

paper.[10]

I will now consider what happens if one wants to maintain reversal of order but wants to weaken some others of the axioms in theorem 1. In particular, the next two sections will weaken independence, and the section after that will weaken WARP.

2.3 Weakening Independence

This and the next section will use an Anscombe-Aumann setting to analyze some models that allow for, or even imply, strict preference for randomization.[11] This makes the task substantially harder because it implies strictly interior choice from some convex sets, thus completeness and transitivity of $\succsim_C$ will not be recovered.[12]

I will first state this section's new axioms – all of which will seem familiar to readers – and the main result. The remainder of the section will be devoted to a discussion that involves some "looking under the hood" to evaluate the result's generalizability. First, some "baseline axioms" are as follows. (Recall the set of acts is now $\mathcal{F}$.)

Axiom 5 ***Monotonicity***

If $f(s) \in C(\Delta\{f(s), g(s)\})$ for all s, then $f \in C(\Delta\{f, g\})$.

Axiom 6 ***Archimedean Property***

For all acts $f \in \mathcal{F}$ and constant acts $p, r \in \Delta\mathcal{X}$:

(i) *If $\lambda p + (1-\lambda)r \succ_C f$ for all $\lambda \in (0,1]$, then not $f \succ_C r$.*

(ii) *If $\lambda p + (1-\lambda)r \prec_C f$ for all $\lambda \in (0,1]$, then not $f \prec_C r$.*

Axiom 7 ***Nontriviality***

$C(M) \subset M$ for some (compact convex) M, where "$\subset$" denotes strict inclusion.

Axiom 8 ***Uncertainty Aversion***

C is convex-valued.

[10] The result can be contrasted with findings by Seo (2009) and Segal (1990). Both consider first-stage randomization as well (where $\mathcal{A} = \Delta\mathcal{X}$ in Segal (1990) and $\mathcal{A} = \mathcal{F}$ in Seo (2009)), do not impose reversal of order, and arrive at representations that imply cardinal utility information over "pure" acts in $\mathcal{A}$. Corollary 2 only recovers a preference ordering, i.e. ordinal such information. The reason lies in the smaller choice domain. Cardinal utility information is revealed through probability trade-offs: If $a \succ b \succ c$, then what randomization over $\{a, c\}$ is as good as b? In the setting of corollary 2, it is fundamentally impossible to elicit this indifference because the decision maker cannot be offered a specific *randomization* over $\{a, c\}$ without being offered all such randomizations, and the corresponding *mixture* of $\{a, c\}$, which can be offered, is not any more assumed equivalent to the randomization.

[11] Klibanoff (2001) characterizes maxmin expected utility and Choquet expected utility in terms of when exactly this strict preference obtains.

[12] For example, Wayne from this paper's introduction reveals $(2,1) \succ_C (1,0) \sim_C (0,0) \sim_C (0,2) \bowtie_C (2,1)$ and $(1,0) \bowtie_C (0,2)$, thus $\succsim_C$ is incomplete and both $\succsim_C$ and $\sim_C$ are intransitive.

Monotonicity is the choice correspondence analog of a standard axiom. The Archimedean property not only inherits the weakenings embedded in the constant act Archimedean property, but introduces a third one: It is only imposed for constant "sandwich acts" (p, r). Nontriviality is self-explanatory. Uncertainty aversion is the choice correspondence version of Schmeidler's (1989) axiom. Given this paper's motivation, it is worth reiterating that uncertainty aversion mandates a weak preference *for* randomization, but also that its intuitive appeal must be evaluated in conjunction with reversal of order.

Next is a familiar sequence of independence-type axioms.

Axiom 9 ***C-Independence***

$$C(\lambda M + (1-\lambda)p) = \lambda C(M) + (1-\lambda)p$$

for any (compact convex) menu $M \subseteq \mathcal{F}$ *and constant act* $p \in \Delta\mathcal{X}$*.*

Axiom 10 ***Comonotonic Independence***

$$C(\lambda M + (1-\lambda)f) = \lambda C(M) + (1-\lambda)f$$

for any (compact convex) menu $M \subseteq \mathcal{F}$ *and act* $f \in \mathcal{F}$ *s.t. any two acts* $g, h \in M \cup \{f\}$ *are pairwise comonotonic (there exists no states* s, t *with* $g(s) \succ_C g(t)$ *but* $h(s) \prec_C h(t)$*).*

Axiom 11 ***Independence***

$$C(\lambda M + (1-\lambda)f) = \lambda C(M) + (1-\lambda)f$$

for any (compact convex) menu $M \subseteq \mathcal{F}$ *and act* $f \in \mathcal{F}$*.*

This section's core finding is as follows.

Theorem 2 *Let* C *be defined on all compact convex menus* $M \subseteq \mathcal{F}$ *and assume throughout that* C *fulfils WARP, nonemptiness, monotonicity, nontriviality, and the Archimedean property. Then:*

(i) *C fulfils c-independence and uncertainty aversion iff it can be written as*

$$C(M) = \arg\max_{f \in \Delta M} \min_{\pi \in \Gamma} \int \int U(x) df(s) d\pi,$$

where U *is as in theorem 1 and* Γ *is a unique nonempty, closed, convex set of finitely additive distributions on* $(\mathcal{S}, \Sigma)$*.*

(ii) *C fulfils comonotonic independence iff it can be written as*

$$C(M) = \arg\max_{f \in \Delta M} \int \int U(x) df(s) d\nu$$

where U is as in theorem 1 and ν is a unique nonadditive probability on (S, Σ).

(iii) *C fulfils independence iff it can be written as*

$$C(M) = \arg\max_{f \in \Delta M} \int \int U(x) df(s) d\pi$$

where U is as in theorem 1 and π is a unique finitely additive probability on (S, Σ).

In words, the theorem provides choice correspondence analogs of Gilboa and Schmeidler's (1989) characterization of maxmin expected utility, Schmeidler's (1989) characterization of Choquet expected utility, and Anscombe and Aumann's (1963) characterization of subjective expected utility. Of course, the result also covers models that use strengthenings of these axioms.

While (iii) resembles theorem 1, several new ideas are needed to establish (i) and (ii). In particular, because $\succsim_C$ is not complete nor transitive, it cannot fulfil Gilboa and Schmeidler's (1989) or Schmeidler's (1989) axioms. The proof rather identifies completions $\succsim$ of $\succsim_C$ that fulfil relevant preference axioms and induce the same choice correspondence.[13] The decision maker can, then, be modeled *as if* she optimized according to $\succsim$.

Implementing this idea involves two steps: First, define $\succsim$ and verify that it extends $\succsim_C$; second, verify that $\succsim$ fulfils specific axioms. The first step is similar between parts (i) and (ii) and may be applicable to more models. Specifically, $\succsim$ is defined as

$$f \succsim g \Leftrightarrow \left[\exists p, q \in \Delta\mathcal{X} : f \sim_C p \succsim_C q \sim_C g\right].$$

That is, $\succsim$ ranks any two acts f and g according to the ranking of their certainty equivalents. To be sure, it is not obvious that each act has a unique certainty equivalent, nor that $\succsim$ induces C as choice correspondence. However, a lemma formally established in the appendix shows that these properties obtain if C fulfils WARP, nonemptiness, monotonicity, the Archimedean property, constant act independence, and furthermore has the feature that $f \bowtie_C p$ for *no* act f and constant act p. Since most of these axioms are imposed in many models, the proof of theorem 2 can likely be adapted whenever this last feature can be verified. In particular, given baseline axioms, the feature follows from either c-independence or comonotonic independence.

Some subtleties in the second proof step are as follows. Convex-valuedness of C implies uncertainty aversion of $\succsim$ only given the Archimedean property. Also, this paper's Archimedean property suffices because it is only used in the first proof step, in particular to conclude the constant act Archimedean property and the existence of certainty equivalents. Once $\succsim$ is defined, c-independence or comonotonic independence (again, together with baseline axioms) imply an Archimedean property of $\succsim$. Finally, some aspects of the Archimedean property's weakening are persistent: Unlike with theorem 1, a more

[13] For precursors to this proof strategy, see in particular Richter's (1966) use of Szpilrajn (1938).

conventional axiom ($f \succ_C g \succ_C h$ implies $\lambda f+(1-\lambda)h \succ_C g \succ_C \gamma f+(1-\gamma)h$ for some $\lambda, \gamma \in (0,1)$) is in general failed.[14] As before, the theorem implies a characterization of the corresponding preferences through axiomatic systems that avoid transitivity and weaken continuity.

2.4 Maximin Utility for Statistical Decisions

For reasons discussed in section 1.2, statistical decision theory is a natural application of this paper's approach. In this section, I focus on a decision criterion that is somewhat specific to statistics, namely maximin utility as originally conceived by Wald (1950). The according choice correspondence is

$$C(M) = \arg\max_{f\in M} \min_{s\in\mathcal{S}} \int U(x)df(s).$$

This contrasts with maxmin expected utility by not using a set of priors.[15] An important difference in the relevant axiomatizations (Milnor (1954), Stoye (2011a)) is that c-independence is not needed other than by implying constant act independence. The crucial aspect of characterizing statistical maximin utility is not the weakening from independence to c-independence, but the introduction of a symmetry axiom (Arrow and Hurwicz (1972)) that can be adapted to the present setting as follows.

Axiom 12 ***Symmetry***

For any menu $M \subseteq \mathcal{F}$, let $E, F \in \Sigma \backslash \{\varnothing, \mathcal{S}\}$ be any two disjoint events s.t. any $f \in M$ is constant on E as well as F. For any $f \in M$, define f' by

$$f'(s) = \begin{cases} f(s)\|_{s\in E}\,, & s \in F \\ f(s)\|_{s\in F}\,, & s \in E \\ f(s) & \text{otherwise} \end{cases}.$$

Then

$$f \in C(M) \Leftrightarrow f' \in C(\{g' : g \in M\}).$$

Symmetry excludes any prior weighting of states, for example by likelihood, and has been discussed in detail in the aforecited references. Replacing c-independence with symmetry causes a difficulty because comparability of any act with any constant act is not any more (immediately) implied, thus the proof technique from theorem 2 fails. Nonetheless, the following result obtains.

Theorem 3 *Let Σ contain at least three distinct events and let C be defined on all compact convex menus $M \subseteq \mathcal{F}$. C fulfils WARP, nonemptiness, constant act independence, monotonicity, the*

[14] Let $f = (3,3)$, $g = (3,2)$, and $h = (0,1)$, then Wayne reveals $f \succ_C g \succ_C h$ but $\lambda f + (1-\lambda)h \succ_C g$ for no $\lambda \in (0,1)$.

[15] Of course, it is technically embedded by corresponding to the maximal set of priors. But if interpreted as a characterization, this observation would miss out on the fact that in their own view, frequentist statisticians fundamentally avoid priors rather than arbitrarily choosing a particularly large set of priors.

Archimedean property, uncertainty aversion, nontriviality, and symmetry iff it can be written as

$$C(M) = \arg\max_{f\in M}\min_{s\in\mathcal{S}}\int U(x)df(s)$$

with U as in theorem 1.

Thus, the observation that c-independence is not needed to characterize the statistical minimax principle continues to apply. Convexifying choice sets requires to re-establish it from scratch, however.

A brief intuition for the proof is as follows. Identifying the constant acts with the real numbers, the certainty equivalent of f is defined as infimum of $\{p : p \succ_C f\}$, and $\succsim$ is defined as before. Substantial adaptation of ideas that go back to Milnor (1954) yields that every act's certainty equivalent is the constant act just dominated by it, thus $\succsim$ is as desired. Finally, it follows that every act dominates its own certainty equivalent, recovering comparability with constant acts (through monotonicity) but also informing a direct proof that $\succsim$ extends $\succsim_C$.

2.5 Weakening WARP: A Characterization of the Bewley Model

An old but recently revived literature examines axiomatic foundations for incomplete preferences.[16] To embed this literature in this paper's approach, one must separate two sources of incompleteness of $\succsim_C$, namely "true" noncomparability of acts and observational noncomparability that arises from randomization. This section provides some steps in this direction by providing a characterization of a multi-prior utility model inspired by Bewley (2002). The specific version that will be axiomatized corresponds to Gilboa, Maccheroni, Marinacci, and Schmeidler's (2010) "objective rationality" preorder:

$$f \succsim g \text{ iff } \int\int U(x)dg(s)d\pi \geq \int\int U(x)df(s)d\pi \text{ for all } \pi\in\Gamma,$$

where U is as before and Γ is a unique compact convex set of priors. Note that $\succsim$ is incomplete unless Γ is a singleton.

Incomplete preferences raise a conceptual question: What does it mean for $\succsim$ to rationalize C? The most common definition in the literature is to say that this obtains if

$$C(M) = \{f \in M : g \succ f \text{ for no } g \in M\}.$$

Two important implications of this definition are as follows. First, for the model considered in this section, C is nonempty-valued but violates WARP, so that the latter must be weakened. Second, $C(\Delta\{f,g\}) = \Delta\{f,g\}$ cannot any more be taken to indicate indifference – it could also be due to "true"

[16] See Aumann (1962) and Bewley (2002) for classic and Dubra, Maccheroni, and Ok (2004), Galaabaatar and Karni (2013), Gilboa, Maccheroni, Marinacci, and Schmeidler (2010), and Ok, Ortoleva, and Riella (2012) for recent contributions. I thank an associate editor for raising the question considered in this section.

noncomparability of f and g. Indeed, observationally separating noncomparability from indifference is the core challenge.

The crucial idea here, which is due to Bandhyopadhay and Sengupta (2003) and Eliaz and Ok (2006), goes as follows. Indifferent acts f and g should have the property that either both or neither are picked whenever both are available. In contrast, if f and g are noncomparable, there should be choice problems in which f is picked "over" g and vice versa. For example, it would seem plausible that if an act h slightly but unambiguously improves on f, then g but not f is sometimes chosen in its presence. These considerations inform the following axiom:

Axiom 13 ***WARNI (Weak Axiom of Revealed Non-Inferiority)***

Say that f blocks g if $f \in M \Rightarrow g \notin C(M)$. Then $f \in M/C(M)$ only if some $g \in C(M)$ blocks f.

WARNI obviously (much) weakens WARP. It is not quite sufficient to guarantee the desired representation. Additional vestiges of WARP will be introduced by strengthening the monotonicity and continuity axioms.

Axiom 14 ***Strengthened Monotonicity***

If $f(s) \in C(\Delta\{f(s), g(s)\})$ for all s, then $g \in C(M) \Rightarrow f \in C(M)$ for all $M \supset \{f, g\}$.

Axiom 15 ***Strengthened Archimedean Property***

Say that f almost blocks g if there exist $h \in \mathcal{F}$ s.t. $g \notin C(\Delta\{(1-\alpha)f + \alpha h, g\})$ for all $\alpha \in (0, 1]$. Then:

(i) *If f almost blocks g, then $g \in C(M) \Rightarrow f \in C(M)$ for all $M \supset \{f, g\}$.*

(ii) *If f almost blocks g but g does not almost block f, then f blocks g.*

The above axioms are fulfilled by models considered so far. In particular, strengthened monotonicity and part (i) of the strengthened Archimedean property are implied by the previous versions of these axioms, together with WARP. They introduce a vestige of WARP because the conclusion is imposed not only with respect to choice from $\Delta\{f, g\}$, but choice from any menu containing $\Delta\{f, g\}$. Regarding part (ii) of the extended Archimedean axiom, the reader might find intriguing that intuitively speaking, the axiom enforces closedness (and not openness) of strict revealed preference along certain paths.[17]

This section's main result is as follows.

Theorem 4 *Let C be defined on all compact convex menus $M \subseteq \mathcal{F}$. C fulfils WARNI, nonemptiness, nontriviality, strengthened monotonicity, and the strengthened Archimedean property iff it can be writ-*

[17] A "cute" example for non-redundancy of part (ii) is this section's model, except that strict preference requires f to be *strictly* better than g under *all* priors, as in Bewley (2002) and Galaabaatar and Karni (2013). (If f is weakly better under all and strictly better under some but not all priors, the two are noncomparable in this example.)

ten as $C(M) = \{f \in M : g \succ f$ *for no* $g \in M\}$*, where* $\succ$ *is the asymmetric part of* $\succsim$ *as defined above, and where properties of* U *and* Γ *are as before.*

Because of the difficulty of interpreting $C(\Delta\{f,g\}) = \Delta\{f,g\}$, the proof of theorem 4 takes a slightly different route: Define $f \succ g$ if f blocks g, $f \sim g$ if either both or neither are picked from any menu containing both, and $\succsim$ as the union of the two. Then $\succsim$, which turns out to correspond to "almost binding," rationalizes C and fulfils the axioms of Gilboa et al. (2010, theorem 1).

I conclude by briefly mentioning two other results whose proofs only marginally expand on the above.

Remark 2 *Gerasimou (2012) defines rationalization as in remark 1, i.e.* $C(M) = \{f \in M : g \in M \Rightarrow f \succsim g\}$*, and accepts that* C *may be empty-valued for some* M*. He provides axioms which ensure that* C *is rationalized in this sense by the preorder* $\succsim_C$*, where* $f \succsim_C g$ *iff* $f \in C(\{f,g\})$*. Embedding his result in this paper's setting and using independence as in the proof of theorem 4, one can show that additional imposition of nontriviality, monotonicity, and an Archimedean property yields a representation in terms of the same* $\succsim$ *defined above. Indeed, Gerasimou's (2012) axioms are sufficiently strong so that, if one takes this route, the monotonicity and Archimedean axioms from previous sections suffice.*

Remark 3 *If* Σ *contains at least three distinct events, then proof steps from theorem 3 can be mimicked to show that the symmetry axiom enforces a maximal set of priors. That is, adding symmetry to this section's axioms characterizes the choice correspondence which only eliminates statewise dominated acts. This result is similar to the adaptation to randomization of one reported in Stoye (2012b).*

3 Conclusion

Whenever agents can randomize, it is of interest to model their behavior in terms of choice from convex sets. WARP then fails to imply completeness and acyclicity of directly revealed preference. I reconsidered several models of decision making under risk or uncertainty in this light, imposing nonemptiness of choice correspondences and WARP; equivalently, I imposed "decisiveness" of preference relations but not their completeness and acyclicity. Characterizations of von Neumann-Morgenstern utility, Anscombe-Aumann subjective expected utility, Gilboa-Schmeidler maxmin expected utility, Choquet expected utility, Bewley's multiple prior model, and Wald-Milnor (statisticians') maximin utility continue to go through. It would be of obvious interest to extend these findings to more models, e.g. the multi-utility models cited in the preceding section or skew-symmetric bilinear utility (Fishburn (1982)), all of which fail the expected utility representation for constant acts.[18]

[18] Stoye (2011b) recovers a certain minimax regret model in a setting where C fails WARP as well as WARNI.

Another interesting extension would be to be even more strict about the decision theoretic environment or about what is considered observable. The present setting assumed objective state spaces and objective randomization, which is plausible in some important applications, notably statistical decision theory and probably also games, but not in all of them. Also, I assumed that randomization over acts is directly observable: If we only offer $\{(2,1),(0,2)\}$ to Wayne, we see him choose the former with probability 2/3. In many settings, including many experimental designs, we would really just observe a realization of his randomization. This adds a layer of measurement error, though it would not seem qualitatively different from measurement error in choice experiments more generally.

Finally, what may be most contentious about the setting is that "hedging by randomization" asks much of agents' ability to commit, and that it turns reversal of order into a strong axiom, especially when combined with ambiguity aversion. Careful consideration of these issues has informed ongoing research and might lead to further, interesting extensions of this paper's approach.

A Proofs

Theorem 1 The strategy is to show that $\succsim_C$ fulfils the axioms of Herstein and Milnor (1953). To see completeness, fix any $p,q \in \Delta\mathcal{X}$. By nonemptiness, there exists $\lambda \in [0,1]$ s.t. $\lambda p + (1-\lambda)q \in C(\Delta\{p,q\})$. If $\lambda = 0$, then $q \succsim_C p$. Else, $\lambda p + (1-\lambda)q \in C(\Delta\{\lambda p + (1-\lambda)q, q\})$ by WARP, hence $p \succsim_C q$ by independence. Next, suppose there exist (p,q,r) s.t. $p \succ_C q \succsim_C r \succsim_C p$. Consider choice from $M = \Delta\{p,q,r\}$. $p \succ_C q$ and independence jointly imply that $\lambda p + (1-\lambda)r \succ_C \lambda q + (1-\lambda)r$ for all $\lambda \in (0,1]$; using independence again (mixing with $\lambda p + (1-\lambda)r$), one finds that $\lambda p + (1-\lambda)r \succ_C (1-\gamma)\lambda p + \gamma\lambda q + (1-\lambda)r$ for any $\gamma, \lambda \in (0,1]$, hence $C(M) \subseteq \Delta\{p,r\}$. Since C is nonempty, there specifically exists $\lambda \in [0,1]$ s.t. $\lambda p + (1-\lambda)r \in C(M)$. This establishes transitivity. Independence (used with r as mixing act) then implies $r \in C(M)$, hence $q \in C(M)$, contradicting $C(M) \subseteq \Delta\{p,r\}$. Independence is immediate. To see continuity, fix any acts $p,q,r \in \Delta\mathcal{X}$; the aim is to show closedness of $\{\lambda : \lambda p + (1-\lambda)q \succsim_C [\precsim_C] r\}$. W.l.o.g. assume $p \succsim_C q$ and consider any $\lambda, \lambda' \in [0,1]$ with $\lambda > \lambda'$, then independence yields $\lambda p + (1-\lambda)q \succsim_C \lambda' p + (1-\lambda')q$. The claim follows immediately if either $q \succsim_C r$ or $r \succsim_C p$ and from the constant act Archimedean property otherwise.

Theorem 2 WARP, nonemptiness, monotonicity, the Archimedean property, and constant act independence (which is implied by all the other independence-type axioms) are assumed throughout. Theorem 1 then applies to constant acts. Define the mapping $u : \Delta\mathcal{X} \to \mathbb{R}$ by $u(p) = \int U(x)dp$, where U is the utility function from theorem 1; thus, $u \circ f(s)$ is the expected utility of act f in state s. Then:

Lemma 5 *Fix any menus M and M' s.t. there exists a one-to-one mapping $(\cdot)' : M \to M'$ from acts in M to acts in M' s.t. $u \circ f = u \circ f'$. Then $C(M') = (C(M))'$. In particular, $u \circ f = u \circ f'$ and*

$u \circ g = u \circ g'$ *imply that* $f \succsim_C g \Leftrightarrow f' \succsim_C g'$.

Proof. Omitted. ■

Next, say that C fulfils constant act comparability if $f \bowtie_C p$ for no act $f \in \mathcal{F}$ and constant act $p \in \Delta\mathcal{X}$. Then:

Lemma 6 *If C fulfils constant act comparability, then there exists a unique complete, transitive, and monotonic ($u \circ f(s) \geq u \circ g(s), \forall s \Rightarrow f \succsim g$) relation $\succsim$ on $\mathcal{F} \times \mathcal{F}$ s.t. $C(M) = \{f \in M : g \in M \Rightarrow f \succsim g\}$. In particular, $\succsim$ can be defined by $f \succsim g \Leftrightarrow [\exists p, q \in \Delta\mathcal{X} : f \sim_C p \succsim_C q \sim_C g]$.*

Proof. Lemma 5 applies, thus one can restrict attention to utility acts $u \circ f$. Monotonicity and nonemptiness imply monotonicity of $\succsim_C$, i.e. $u \circ f \geq u \circ g \Rightarrow f \succsim_C g$. For simplicity, I henceforth write $f = g$ for $u \circ f = u \circ g$ and leave simple uses of WARP and monotonicity implicit.

Step 1: Every completion $\succsim$ of $\succsim_C$ induces choice correspondence C. To see this, first assume $f \in C(M)$, then $g \in M \Rightarrow f \succsim_C g \Rightarrow f \succsim g$. Now assume $f \in M \backslash C(M)$, then by nonemptiness, there exists $g \in M$ with $g \succsim_C f$. If $g \sim_C f$, then $C(M) \cap \{f, g\} = \{g\}$ would contradict WARP, hence $g \succ_C f \Rightarrow g \succ f$.

Step 2: Every act f is revealed indifferent to exactly one constant act, henceforth called its certainty equivalent and denoted $c(f)$. Furthermore, $p \succsim_C f \Leftrightarrow p \succsim_C c(f)$ for any act f and constant act p. To establish this, first observe that $f \succsim_C p \succsim_C q \Rightarrow f \succsim_C q$ and $f \succsim_C p \succ_C q \Rightarrow f \succ_C q$. To see this, assume $f \succsim_C p \succsim_C q$ and write

$$\begin{aligned} & C(\Delta\{f, p, q\}) \cap \Delta\{f, p\} = C(\Delta\{f, p\}) \text{ (by monotonicity, WARP, } p \succsim_C q \text{, and nonemptiness)} \\ \Longrightarrow \quad & f \in C(\Delta\{f, p, q\}) \text{ (by } f \succsim_C p), \end{aligned}$$

implying the first claim by WARP. The second claim follows because if furthermore $p \succ_C q \Rightarrow q \notin C(\Delta\{f, p, q\}) \Rightarrow q \notin C(\Delta\{f, q\})$ by repeated uses of WARP and $f \in C(\Delta\{f, p, q\})$.

Next, $p \succsim_C q \succsim_C f \Rightarrow p \succsim_C f$ and $p \succ_C q \succsim_C f \Rightarrow p \succ_C f$. To see this, note that $C(\Delta\{f, p, q\}) \cap \Delta\{f, p\} = C(\Delta\{f, p\})$ (shown before) implies $C(\Delta\{f, p, q\}) \cap \{f, p\} \neq \varnothing$ by constant act comparability. But $f \in C(\Delta\{f, p, q\}) \Rightarrow q \in C(\Delta\{f, p, q\}) \Rightarrow p \in C(\Delta\{f, p, q\})$ by WARP, thus $p \in C(\Delta\{f, p, q\})$. Together, these facts imply the existence of *at most* one certainty equivalent of f and also that $p \succsim_C f \Leftrightarrow p \succsim_C c(f)$ if $c(f)$ exists.

To see existence of *at least* one certainty equivalent, let the constant acts $\underline{f}$ and $\overline{f}$ be s.t. $u \circ \underline{f} = \min_{s \in \mathcal{S}} u \circ f(s)$ and $u \circ \overline{f} = \max_{s \in \mathcal{S}} u \circ f(s)$, then $\overline{f} \succsim_C f \succsim_C \underline{f}$ by monotonicity. If either preference is weak, $c(f)$ has been discovered. Else, let $\lambda^* = \inf\{\lambda \in (0,1) : \lambda\overline{f} + (1-\lambda)\underline{f} \succ_C f\}$, then constant act comparability, the facts previously established in this step, and the Archimedean property jointly imply $\lambda^*\overline{f} + (1-\lambda^*)\underline{f} \sim_C f$.

Step 3: Define $\succsim$ by $f \succsim g \Leftrightarrow c(f) \succsim_C c(g)$. This ordering is complete and transitive by theorem 1. Next, $\succsim$ extends $\succsim_C$. To see this, let $f \succsim_C g$ and consider choice from $\Delta\{f, g, c(f)\}$. Suppose by contradiction that $c(f) \notin C(\Delta\{f, g, c(f)\})$, hence $C(\Delta\{f, g, c(f)\})$ contains some $h = \alpha f + \beta g + (1 - \alpha - \beta)c(f)$, where $\alpha + \beta > 0$. Define $i = (\alpha/(\alpha + \beta))f + (\beta/(\alpha + \beta))g$. Noting that $h \in \Delta\{c(f), i\}$ and $i \in \Delta\{f, g\}$, WARP and constant act comparability can be used to write

$$h \in C(\Delta\{f, g, c(f)\}) \Longrightarrow h \in C(\Delta\{c(f), i\}) \Longrightarrow c(f) \notin C(\Delta\{c(f), i\}) \Longrightarrow i \in C(\Delta\{c(f), i\})$$
$$\Longrightarrow i \in C(\Delta\{f, g, c(f)\}) \Longrightarrow i \in C(\Delta\{f, g\}) \Longrightarrow f \in C(\Delta\{f, g, c(f)\}) \Longrightarrow c(f) \in C(\Delta\{f, g, c(f)\}),$$

a contradiction. Also using step 2, one finds

$$c(f) \in C(\Delta\{f, g, c(f)\}) \Longrightarrow c(f) \succsim_C g \Longrightarrow f \succsim g.$$

Finally, WARP and monotonicity of $\succsim_C$ now yield monotonicity of $\succsim$. ■

We are now in a position to prove the main results. First, all of the independence-type axioms imply constant act comparability. To see this for c-independence, fix any act f and constant act p, then $\lambda f + (1 - \lambda)p \in C(\Delta\{f, p\})$ for some $\lambda \in [0, 1]$ by nonemptiness. If $\lambda = 0$, then $p \succsim_C f$. Else, WARP yields $\lambda f + (1 - \lambda)p \in C(\Delta\{\lambda f + (1 - \lambda)p, p\})$, and c-independence then yields $f \succsim_C p$. The argument for the other axioms is similar. Thus, lemma 6 applies, and it suffices to show that $\succsim$ defined there fulfils relevant preference axioms. I only explicitly show that axioms used in (i) imply Gilboa and Schmeidler's (1989) axioms; the other cases are easier.

Completeness, transitivity, monotonicity, and nontriviality are clear. C-independence of C implies

$$f \sim_C c(f) \Longleftrightarrow \lambda f + (1 - \lambda)p \sim_C \lambda c(f) + (1 - \lambda)p \Longleftrightarrow c\left(\lambda f + (1 - \lambda)p\right) = \lambda c(f) + (1 - \lambda)p$$

for any constant act p (using uniqueness of certainty equivalents). One can then write

$$c(f) \succsim_C c(g) \Longleftrightarrow \lambda c(f) + (1 - \lambda)p \succsim_C \lambda c(g) + (1 - \lambda)p \Longleftrightarrow c\left(\lambda f + (1 - \lambda)p\right) \succsim_C c\left(\lambda g + (1 - \lambda)p\right),$$

hence $\succsim$ is c-independent.

Define the function $I : \mathcal{F} \to \mathbb{R}$ by identifying $I(f)$ with the value of $u \circ c(f)$, then I is continuous on $\Delta\{f, g\}$ for any acts f, g. To see this, fix f, g, some number $\lambda \in (0, 1)$, and consider the act $(1 - \lambda)f + \lambda g$. Let $\underline{g}$ $[\overline{g}]$ be the constant act with utility value $\min_{s \in \mathcal{S}} u \circ g(s)$ $[\max_{s \in \mathcal{S}} u \circ g(s)]$, then numerous uses of this proof's earlier insights, the expected utility representation for constant acts, and monotonicity of $\succsim$ yield

$$(1 - \lambda)c(f) + \lambda \underline{g} \sim (1 - \lambda)f + \lambda \underline{g} \precsim (1 - \lambda)f + \lambda g \precsim (1 - \lambda)f + \lambda \overline{g} \sim (1 - \lambda)c(f) + \lambda \overline{g}$$
$$\Longrightarrow I\left((1 - \lambda)c(f) + \lambda \underline{g}\right) \leq I\left((1 - \lambda)f + \lambda g\right) \leq I\left((1 - \lambda)c(f) + \lambda \overline{g}\right)$$
$$\Longrightarrow (1 - \lambda)I(f) + \lambda I(\underline{g}) \leq I\left((1 - \lambda)f + \lambda g\right) \leq (1 - \lambda)I(f) + \lambda I(\overline{g}),$$

thus I is continuous at f. The argument can be mimicked for any act on $\Delta\{f,g\}$.

Hence, $\succsim$ fulfils a conventional Archimedean property for preferences. Furthermore, consider uncertainty aversion, i.e. the claim that $f \sim g$ implies $\lambda f + (1-\lambda)g \succsim f$ for all $\lambda \in (0,1)$. Suppose this fails, then there exist f, g s.t. $c(f) = c(g)$ and $\lambda^* \in (0,1)$ s.t. $I(\lambda^* f + (1-\lambda^*)g) < I(f)$. Fix any $u^* \in (I(\lambda^* f + (1-\lambda^*)g), I(f))$. By continuity of I, $\underline{\lambda} = \max\{\lambda \in [0,\lambda^*] : I(\lambda f + (1-\lambda)g) \geq u^*\}$ and $\overline{\lambda} = \min\{\lambda \in [\lambda^*,1] : I(\lambda f + (1-\lambda)g) \geq u^*\}$ exist and $I(\underline{\lambda} f + (1-\underline{\lambda})g) = I(\overline{\lambda} f + (1-\overline{\lambda})g) = u^*$. But this implies $C(\Delta\{\underline{\lambda} f + (1-\underline{\lambda})g, \overline{\lambda} f + (1-\overline{\lambda})g\}) = \{\underline{\lambda} f + (1-\underline{\lambda})g, \overline{\lambda} f + (1-\overline{\lambda})g\}$, contradicting convex-valuedness of C.

Theorem 3

Step 1: Preliminaries. Lemma 5 applies, thus one can identify acts with utility acts. Also, for any act f, let $\underline{f}$ $[\overline{f}]$ be the constant act with utility value $\min_{s\in\mathcal{S}}[\max_{s\in\mathcal{S}}]u \circ f(s)$. Then for any constant act p, $p \succ_C \overline{f} \Rightarrow p \succ_C f$ and $\underline{f} \succ_C p \Rightarrow f \succ_C p$. To see the former, note that $p \succsim_C f$ by monotonicity and nonemptiness. Suppose $p \sim_C f$, then WARP and monotonicity would imply that $\{p, f, \overline{f}\} \subseteq C(\Delta\{p, f, \overline{f}\})$ and hence $p \sim_C \overline{f}$, a contradiction. The other argument is similar.

Noting that the constant acts are isomorphic with (a subset of) the real numbers, define the certainty equivalent $c(f)$ of any act f as $c(f) = \inf\{p : p \succ_C f\}$. As before, define the complete and transitive relation $\succsim$ on $\mathcal{F} \times \mathcal{F}$ by $f \succsim g \Leftrightarrow c(f) \succsim_C c(g)$. Step 1 of lemma 6 applies, so the strategy will be to characterize $\succsim$ and then argue that it extends $\succsim_C$. Recall that $c(p) = p$ for any constant act p and that $\succsim$ is monotonic.

Step 2: Characterizing Certainty Equivalents. Fix any Σ-measurable partition of $\mathcal{S}$ into three nonempty events $\{E_1, E_2, E_3\}$ and temporarily restrict attention to acts that are constant on elements of $\{E_1, E_2, E_3\}$ and that will be identified with utility vectors $(u_1, u_2, u_3) \in \mathbb{R}^3$. Fix any scalars $u > v$ within the range of U. Then one can make the following observations.

First, $((u+v)/2, (u+v)/2, v) \succsim (u, v, v)$. To see this, it suffices to show $p \succ_C ((u+v)/2, (u+v)/2, v) \Rightarrow p \succ_C (u, v, v)$. Thus, fix p and consider choice from $M = \Delta\{p, (u,v,v), (v,u,v)\}$. As M is invariant under exchange of the consequences of the first two events, WARP, nonemptiness, symmetry, and uncertainty aversion jointly imply that $C(M) \cap \Delta\{p, ((u+v)/2, (u+v)/2, v)\} = C(\Delta\{p, ((u+v)/2, (u+v)/2, v)\})$ and also that $(u,v,v) \in C(M) \Rightarrow ((u+v)/2, (u+v)/2, v) \in C(M)$. Thus, write

$$p \succ_C ((u+v)/2, (u+v)/2, v) \Longrightarrow C(M) \cap \{p, ((u+v)/2, (u+v)/2, v)\} = \{p\}$$
$$\Longrightarrow C(M) \cap \{p, (u,v,v)\} = \{p\} \Longrightarrow p \succ_C (u,v,v),$$

where the last step uses WARP. One can now write

$$((u+v)/2,(u+v)/2,v) \succsim (u,v,v) \sim (v,v,u) \sim (u,u,v) \succsim ((u+v)/2,(u+v)/2,v).$$

Here, $(u,v,v) \sim (v,v,u)$ follows because $p \succ_C (u,v,v) \Leftrightarrow p \succ_C (v,v,u)$ by symmetry, and $(v,v,u) \sim (u,u,v)$ follows similarly. Hence, $(u,u,v) \sim ((u+v)/2,(u+v)/2,v)$. Iterating the argument and using transitivity and monotonicity of $\succsim$, it follows that $(u,u,v) \sim (w,w,v)$ for any $w > v$. Suppose by contradiction that $c(u,u,v) = (y,y,y)$ for some $y > v$. Then $c((v+y)/3,(v+y)/3,v) = (y,y,y)$, but $((v+y)/2,(v+y)/2,(v+y)/2) \succ_C ((v+y)/3,(v+y)/3,v)$ by step 1, a contradiction. On the other hand, $(u,u,v) \succ_C p$ for any $p \prec_C (v,v,v)$ by step 1. It follows that $c(u,u,v) = (v,v,v)$. Recall this holds for any (u,v) and (E_1,E_2,E_3) subject to conditions stated earlier in this step.

Consider now any nonconstant act f. Let E denote the event on which $u \circ f(s) = u \circ \underline{f}$ and write $f_E g$ for the act that agrees with f on E and with g otherwise, then $\underline{f}_E\overline{f} \sim \underline{f}$ by the preceding paragraph's result. On the other hand, $\underline{f}_E\overline{f} \succsim f \succsim \underline{f}$ by monotonicity of $\succsim$. It follows that $f \sim \underline{f}$, i.e. $\succsim$ is as desired. Finally, monotonicity and nonemptiness imply $f \succsim_C \underline{f}$ and step 1 implies that $f \succ_C p$ for any $p \prec_C \underline{f}$, thus the Archimedean property yields $f \sim_C \underline{f}$.

Step 3: Extension. To see that $\succsim$ extends $\succsim_C$, fix any acts f and g. Suppose $f \succsim_C g$. Consider choice from $\Delta\{f,g,\underline{f},\underline{g}\}$. As f and g dominate their respective certainty equivalents, WARP, nonemptiness, lemma 4, and monotonicity yield $C(\Delta\{f,g,\underline{f},\underline{g}\}) \cap \Delta\{f,g\} = C(\Delta\{f,g\})$, hence $\underline{f} \in C(\Delta\{f,g,\underline{f},\underline{g}\})$, hence $\underline{f} \succsim_C \underline{g}$.

Theorem 4 Define $f \succ g$ if f blocks g. Define $f \sim g$ if for all menus $M \supseteq \{f,g\}$, $f \in C(M)$ iff $g \in C(M)$. Consideration of binary menus shows that $\succ$ and $\sim$ are disjoint. Define $\succsim$ as union of $\succ$ and $\sim$. Then WARNI easily implies that $\succsim$ rationalizes C. Also, I will freely use that WARNI implies $C(M \cup N) \cap M \subseteq C(M)$ (i.e. "property α").

Step 1: $C(\Delta\{f,g\}) \cap \{f,g\} \neq \varnothing$. Suppose otherwise and let $\alpha \in (0,1)$ be s.t. $h = \alpha f + (1-\alpha) g \in C(\Delta\{f,g\})$. Then $h \in C(\Delta\{f,h\})$ by WARNI. But $g \notin C(\Delta\{f,g\})$ and independence, used with f as mixture act and α as mixture probability, jointly imply $h \notin C(\Delta\{f,h\})$.

Step 2: Transitivity and reflexivity of $\succsim$. To see transitivity of $\succ$, suppose $f \succ g \succ h$ and consider choice from $\Delta\{f,g,h\}$. Suppose by contradiction that $C(\Delta\{f,g,h\})$ contains some $j \neq f$. Then there exist $\alpha \in (0,1]$ and $\beta \in [0,1]$ s.t. $j = \alpha f + (1-\alpha)[\beta g + (1-\beta)h]$. WARNI and independence then yield $\beta g + (1-\beta)h \in C(\Delta\{f,g,h\})$, a contradiction because g is blocked by f and any act $\beta g + (1-\beta)h$ for $\beta \in (0,1]$ is blocked by g (the latter claim also uses independence in analogy to step 1). Conclude that $C(\Delta\{f,g,h\}) = \{f\}$, hence f blocks h. Next, $f \sim g$ iff f and g are blocked by the same acts, implying reflexivity and transitivity of $\sim$. Here, "if" is obvious. To see "only if," let $f \sim g$ and suppose that $h \succ f$. Consider choice from $\Delta\{f,g,h\}$, then f cannot be chosen because it is blocked by h, hence g cannot be chosen either, hence some $j \in C(\Delta\{f,g,h\})$ blocks g. Now independence can be used to conclude that some $k \in \Delta\{f,h\}$ blocks g. But $h \succ f$ and independence imply $h \succ k$, hence $h \succ g$ by transitivity. Obviously f and g can be interchanged in this argument.

Step 3: Expected Utility for constant acts. Restrict attention to constant acts and fix a menu M and acts $p, q \in M$. If $\{p,q\} \subseteq C(M)$, then $\{p,q\} \subseteq C(\Delta\{p,q\})$ by WARNI, hence $p \in C(N) \Leftrightarrow q \in C(N)$ for any $N \supset \{p,q\}$ by monotonicity. This establishes WARP for constant acts, hence $\succsim$ is expected utility on the constant acts. In particular, $\succsim$ is c-complete. Monotonicity now also implies that acts are fully summarized by "utility acts."

Step 4: Verification of remaining axioms in GMMS. Independence is obvious. Let f and g fulfil the hypothesis of the monotonicity axiom, then any act that blocks f also blocks g, which easily implies that f almost blocks g. If g almost blocks f as well, then part (i) of strengthened continuity implies that no act can block one but not the other of $\{f,g\}$; thus, $f \sim g$ from step 2. Else, $f \succ g$ by part (ii) of strengthened continuity. To see continuity, assume $\alpha f + (1-\alpha)h \succsim g$ for all $\alpha \in (0,1]$. Assume that indifference holds for two values $\alpha < \alpha'$. Then $\alpha f + (1-\alpha)h \sim \alpha' f + (1-\alpha')h$, but now independence implies that $f \sim \alpha' f + (1-\alpha')h$, hence $f \sim g$ by transitivity. Assume now that indifference holds for at most one value of α, then it is w.l.o.g. (by possibly redefining the f in the axiom's hypothesis) to assume that $\alpha f + (1-\alpha)h \succ g$ for all $\alpha \in (0,1]$. Thus, f almost blocks g, hence $f \succsim g$ as in the argument for monotonicity.

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