Dominance and Admissibility without Priors

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Abstract

This note axiomatizes the incomplete preference ordering that reflects statewise dominance with respect to expected utility, as well as the according choice correspondence. The main motivation is to clarify how *admissibility* as understood by statisticians relates to existing axiomatizations. The answer is that it is characterized by Anscombe and Aumann's (1963) axioms, plus symmetry (Arrow and Hurwicz (1972)), less completeness. Characterizing the according choice correspondence requires relaxing the weak axiom of revealed preference.

Keywords: admissibility, incompleteness, ambiguity, statistical decision theory.

JEL codes: C44, D81.

1 Introduction

This note provides an axiomatic characterization of the incomplete preference ordering described by

$$f \succsim g \Longleftrightarrow \left[\int U(x)df(s) \geq \int U(x)dg(s), \forall s \in \mathcal{S}\right] \tag{1}$$

as well as the corresponding choice rule

$$C(M) = \{f \in M : g \succ f \text{ for no } g \in M\}. \tag{2}$$

Here, $\succ$ is the asymmetric component of $\succsim$, f and g are lottery-acts inducing objective probability distributions $f(s)$ and $g(s)$ over outcomes x in state of the world $s \in \mathcal{S}$, U is a utility function, and a menu M is a set of acts. In words, f is preferred over g if it statewise dominates g in terms of expected utility.

The motivation for this investigation is that $\succsim$ corresponds to dominance of risk functions and membership in C to *admissibility* of a decision rule in the sense that these words are used in statistics.[1]

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[1] See Ferguson (1967) for a classic textbook account.

At the same time, different subsets of the axioms used are known to characterize numerous complete preferences, notably expected utility, maximin utility, and minimax regret. These axioms cannot jointly be fulfilled by a complete preference, yet they characterize $\succsim$ upon dropping completeness. Thinking of completeness as a pragmatic requirement rather than a substantive condition of rationality, one could, therefore, say that admissibility exhausts the overlap between many reasonable decision rules in a precise axiomatic sense.[2] Extending the characterization to C requires additional work because C does not fulfil the weak axiom of revealed preference (WARP, defined precisely below) and it is *not* true that $f \succsim g$ iff f is chosen from $\{f, g\}$.

The next section presents and discusses the main result for $\succsim$, and section 3 extends the characterization to C. Relations to the literature are explained along the way.

2 Main Result

Consider a set $\mathcal{X}$ of ultimate outcomes x, the set $\Delta\mathcal{X}$ of finite lotteries $p, q, \dots$ over such outcomes, and a state space $\mathcal{S}$ with typical element s and sigma-algebra Σ. Acts $f, g, h, \dots \in \mathcal{F}$ are finite, Σ-measurable functions from $\mathcal{S}$ into $\Delta\mathcal{X}$ that map states s onto lotteries $f(s) \in \Delta\mathcal{X}$. An act is *constant* if $f(s)$ does not depend on s. The only restriction on $\mathcal{S}$ is that it must contain three distinct states. With the usual abuse of notation, $\Delta\mathcal{X}$ will be embedded in $\mathcal{F}$ by identifying constant acts with the corresponding lotteries. Mixtures between acts are defined as statewise probabilistic mixtures: $(\lambda f + (1-\lambda)g)(s) = \lambda f(s) + (1-\lambda)g(s)$. Attention will initially be on a preference relation $\succsim$ on $\mathcal{F} \times \mathcal{F}$ and on the following axioms.

P.1: Transitivity

$$f \succsim g \succsim h \Longrightarrow f \succsim h.$$

P.2: Completeness on Constant Acts

$$p \succsim q \text{ or } q \succsim p$$

for all constant acts $p, q \in \Delta\mathcal{X}$.

P.3: Monotonicity

If $f(s) \succsim g(s)$ for all $s \in \mathcal{S}$, then $f \succsim g$.

P.4: Independence

$$f \succsim g \Longleftrightarrow \lambda f + (1-\lambda)h \succsim \lambda g + (1-\lambda)h$$

[2] Indeed, this paper was spawned by the observation that column 1 in table 1 of Stoye (2011b) is not provided in the literature. See the same survey for more background, interpretation, and references.

for all acts f, g, h and scalars $\lambda \in (0,1)$.

P.5: Continuity

There are no acts f, g, h s.t. $\lambda f + (1-\lambda)h \succ g$ for all $\lambda \in [0,1)$ but $f \prec g$ or vice versa.

P.6: Nontriviality

$f \succ g$ for some acts f, g.

P.7: Symmetry

For any acts f, g and disjoint, nonempty events $E, F \in \Sigma$ s.t. both f and g are constant on E as well as F,

$$f \succsim g \Longleftrightarrow f' \succsim g',$$

where f' is defined by

$$f'(s) = \begin{cases} f(s)\|_{s \in E}, & s \in F \\ f(s)\|_{s \in F}, & s \in E \\ f(s) & \text{otherwise} \end{cases}$$

and g' is defined analogously.

P.1-6 are Anscombe and Aumann's (1963) axioms with two modifications: Completeness is only imposed on constant acts, and continuity is weakened along the lines of Aumann (1962). One could strengthen P.5 to mixture continuity, i.e. closedness of weak upper contour sets, without affecting the result. The improvement reported here will be important in the next section, however. Moreover, without completeness, mixture continuity is *not* equivalent to openness of strict upper contour sets. The latter will actually be failed, as will the usual Archimedean axiom ($f \succ g \succ h \Rightarrow \lambda f + (1-\lambda)h \succ g \succ \gamma f + (1-\gamma)h$ for some $\lambda, \gamma \in (0,1)$). Symmetry (Arrow and Hurwicz (1972)) is designed to prevent any imposition of prior weighting among states. It is plausible if, and only if, no prior information about differential plausibility etc. of states is available.

The first main result is as follows:

Theorem 1 *A preference $\succsim$ fulfils axioms P.1-7 iff it can be expressed as in (1), where $U : \mathcal{X} \rightarrow \mathbb{R}$ is unique up to positive affine transformation.*

Proof. I only show "only if." First, the restriction of $\succsim$ to constant acts $p, q \in \Delta\mathcal{X}$ is von Neumann-Morgenstern expected utility by Herstein and Milnor (1953), thus there exists U with claimed properties s.t. $p \succsim q \Leftrightarrow \int U(x)p(x)dx \geq \int U(x)q(x)dx$.[3] Next, define the mapping ("utility act") $u \circ f : \mathcal{S} \rightarrow \mathbb{R}$ by $u \circ f(s) = \int U(x)df(s)$. Then $u \circ f(s) \geq u \circ g(s)$ for all s implies $f \succsim g$ by monotonicity. Also

[3] Mixture continuity, used by Herstein and Milnor (1953), can be derived from solvability, completeness, and independence. But in their first use (their theorem 1), P.5 will obviously do, and the second use (their theorem 2) is redundant given the stronger independence axiom used here.

using transitivity, $u \circ f = u \circ g$ implies $f \sim g$ as well as $f \succsim h \Leftrightarrow g \succsim h$ for all h; thus, utility acts define equivalence classes among acts. I will henceforth equate acts with utility acts. Nontriviality now implies that the range of U is not a singleton, thus it can be normalized to contain $\{-1, 1\}$. Every Σ-measurable step function $u : \mathcal{S} \to [-1, 1]$ then corresponds to an existing (utility) act. Let p_0 be the constant act with zero utility. Independence, used with p_0 as mixing act, implies that $f \succsim g \Leftrightarrow \lambda f \succsim \lambda g$ for every $\lambda \in (0, 1)$. If the range of U is bounded, this suggests an obvious (namely the homothetic) extension of $\succsim$ to the set of Σ-measurable functions $u : \mathcal{S} \to \mathbb{R}$, and the remainder of this proof can be carried out on the extension first. I will henceforth presume that the range of U is unbounded.

Define $f \ominus g$ by $u \circ (f \ominus g)(s) = u \circ f(s) - u \circ g(s)$, then repeated uses of independence (using $-g$ and p_0 as mixture acts) yield $f \succsim g \Leftrightarrow f \ominus g \succsim p_0$. Consider any nonconstant act f, define $\underline{f}$ $[\overline{f}]$ as the constant act with utility value $\min_{s \in \mathcal{S}} u \circ f(s)$ $[\max_{s \in \mathcal{S}} u \circ f(s)]$, let $\underline{E}$ $[\overline{E}]$ be the event on which this utility is achieved, and write $f_E g$ for the act that agrees with f on event E and with g otherwise. Then $\underline{f}_{\underline{E}} \overline{f} \succsim f \succsim \overline{f}_{\overline{E}} \underline{f}$ by monotonicity. Yet also $\underline{f}_{\underline{E}} \overline{f} \succsim p_0 \Leftrightarrow \overline{f}_{\underline{E}} \underline{f} \succsim p_0 \Leftrightarrow \overline{f}_{\overline{E}} \underline{f} \succsim p_0$ by symmetry. Also using independence, monotonicity, and transitivity, it follows that there exist numbers $\alpha, \beta \geq 0$ s.t. $\alpha \min_{s \in \mathcal{S}} u \circ f(s) + \beta \max_{s \in \mathcal{S}} u \circ f(s) > 0$ implies $f \succ p_0$ and $\alpha \min_{s \in \mathcal{S}} u \circ f(s) + \beta \max_{s \in \mathcal{S}} u \circ f(s) < 0$ implies that *not* $f \succsim p_0$. Suppose by contradiction that $\beta > 0$. Let the constant acts (p, q) have utility values $(1 + 3\alpha/\beta, -2)$ and let $\{E, F, G\}$ partition $\mathcal{S}$ into nonempty events (existence of three distinct states ensures feasibility of this), then it would follow that $p_F q \succ p_0$, $p_G q \succ p_0$, but not $\frac{1}{2} p_F q + \frac{1}{2} p_G q \succsim p_0$. This is a contradiction because $p_F q \succ p_0 \Rightarrow \frac{1}{2} p_F q + \frac{1}{2} p_G q \succ \frac{1}{2} p_0 + \frac{1}{2} p_G q$ and $p_G q \succ p_0 \Rightarrow \frac{1}{2} p_0 + \frac{1}{2} p_G q \succ p_0$ by independence, thus $\frac{1}{2} p_F q + \frac{1}{2} p_G q \succ p_0$ by transitivity. Thus $\beta = 0$, hence $\min_{s \in \mathcal{S}} u \circ f(s) \geq 0$ is necessary for $f \succsim p_0$; it is sufficient by monotonicity. ■

This model can be contrasted with a multiple prior version whereby $f \succsim g$ iff $\int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi$ for all priors π in a convex set Γ, as recently axiomatized by Gilboa et al. (2010). The present, "prior-less" version corresponds to the case where Γ collects all distributions on $(\mathcal{S}, \Sigma)$. Gilboa et al.'s (2010) axioms imply P.1-P.5, thus theorem 1 establishes that adding symmetry to them enforces the maximal set of priors. However, the above proof is brief, self-contained, elementary, and weakens continuity. In earlier work, Bewley (2002) uses a finite state space and takes strict preference as primitive, leading to some differences in axioms and results. In particular, f must be *strictly* better than g under *all* priors to be strictly preferred to g, and all priors $\pi \in \Pi$ must have full support; thus, this paper's model is not embedded. Stoye (2011a) axiomatizes a ranking that resembles $\succsim$ in not using priors but resembles Bewley's in that strict preference requires everywhere strict dominance. In addition, the ranking is complete; loosely speaking, noncomparabilities are replaced with indifferences, and this is axiomatized by relaxing transitivity.

3 Extension to Choice Correspondences

A "revealed preference" perspective suggests choices as primitive objects of analysis because they are directly observable. Thus, consider a choice correspondence C that maps finite menus $M \subset \mathcal{F}$ onto nonempty choice sets $C(M) \subseteq M$ and consider imposing axioms on C rather than $\succsim$. Well known consistency conditions for C are:

WARP (Arrow (1959)[4])

$C(M \cup N) \cap M \in \{C(M), \varnothing\}$ for all menus M, N.

Property α (Chernoff (1954), Sen (1969))

$C(M \cup N) \cap M \subseteq C(M)$ for all menus M, N.

Imposing WARP would lead to an immediate dualism between choice and preference (Arrow (1959)): Define the preference revealed by C as "$f \trianglerighteq_C g$ if $f \in C(\{f, g\})$" and the choice correspondence induced by a preference relation $\trianglerighteq$ as $C_{\trianglerighteq}(M) = \{f \in M : g \in M \Rightarrow f \trianglerighteq g\}$, then WARP implies that $\trianglerighteq_C$ is transitive and that $C_{\trianglerighteq_C} = C$. This all fails here: The C in (2) fulfils property α (which is listed for subsequent use) but fails WARP, $\trianglerighteq_C$ is intransitive, and $\succsim \neq \trianglerighteq_{C_{\succsim}}$. To characterize C anyway, one needs the following axioms.

C.1: Admissibility

If $f, g \in M$ and $f \in C(\{f(s), g(s)\})$ for all s, then $g \in C(M) \Rightarrow f \in C(M)$.

C.2: Strict Admissibility

If $f, g \in M$, $f \in C(\{f(s), g(s)\})$ for all s, and $C(\{f(s), g(s)\}) = \{f(s)\}$ for some s, then $g \notin C(M)$.

C.3: Weak Axiom of Revealed Non-Inferiority

(WARNI; Bandhyopadhay and Sengupta (1993), Eliaz and Ok (2006))

Say that g *globally blocks* f if $g \in M \Rightarrow f \notin C(M)$ for all M. Then for all M, $f \in M/C(M)$ implies that some $g \in C(M)$ globally blocks f.

C.4: Independence

$$C(\lambda M + (1-\lambda)f) = \lambda C(M) + (1-\lambda)f$$

for all menus M, acts f, and scalars $\lambda \in (0,1)$, where

$$\lambda M + (1-\lambda)f = \{h \in \mathcal{F} : h = \lambda g + (1-\lambda)f, g \in M\}.$$

C.5: Continuity

If $C(\{f, h\}) = \{f\}$ and $C(\{g, h\}) = \{h\}$, then there exists $\lambda \in (0,1)$ with $C(\{\lambda f + (1-\lambda)g, h\}) = \{\lambda f + (1-\lambda)g, h\}$.

[4] I avoid Arrow's label ("Independence of Irrelevant Alternatives") for terminological consistency with WARNI and to avoid confusion with social choice theory.

C.6: Nontriviality

There exists M s.t. $C(M) \neq M$.

C.7: Symmetry

Fix any menu M and disjoint, nonempty events $E, F \in \Sigma$ s.t. any $f \in M$ is constant on E as well as F. Define f' as before. Then $f \in C(M) \Leftrightarrow f' \in C(\{g' : g \in M\})$.

Axioms C.4-7 much resemble P.4-7. Weakening continuity in analogy to axiom P.5 is now crucial because C is not upper hemicontinuous, and $\trianglerighteq_C$ fails all standard continuity axioms. Axioms C.1-3 are less similar to P.1-3 because it is intricate to choice theoretically distinguish indifference from non-comparability. For example, $\trianglerighteq_C$ conflates the two, rendering it complete yet intransitive. Indeed, the strengthening of monotonicity in axiom C.2 compensates for an inability to directly impose transitivity; it is this axiom (and not transitivity as before) that eliminates the prior-less version of Bewley's (2002) model. Note also that (as shown in the proof) axioms C.1 and C.3 jointly imply WARP for constant acts and that WARNI is easily verified to imply property α.

This section's technical result is as follows.[5]

Theorem 2 *C fulfils axioms C.1-6 iff it can be written as in (2), with $\succsim$ as in theorem 1.*

Proof. Initially restrict attention to menus of constant acts $M \subset \Delta\mathcal{X}$. Then WARP is implied: For any $M, N \subset \Delta\mathcal{X}$, $C(M \cup N) \cap M \subseteq C(M)$ by the property α implication of WARNI. If $C(M \cup N) \cap M \notin \{C(M), \varnothing\}$, then there exist $p \in C(M \cup N)$ and $q \in C(M)/C(M \cup N)$. Property α applied to $C(M)$ yields $C(\{p, q\}) = \{p, q\}$, contradicting axiom C.1 applied to $M \cup N$. Combining Arrow (1959) and Herstein and Milnor (1953), it now follows that C is rationalized by expected utility maximization.

On the general domain, let $f \succ g$ if $C(\{f, g\}) = \{f\}$. Then $\succ$ rationalizes C in the sense of (2). To see this, fix any menu M and act $f \in M$. If $f \notin C(M)$, then some $g \in C(M)$ globally blocks f. The definition of global blocking then implies $g \succ_C f$. If $f \in C(M)$, then $g \succ_C f$ for no $g \in M$ because any such g would block f globally and, therefore, in M. Next, $\succ$ is transitive: Let $C(\{f, g\}) = \{f\}$ and $C(\{g, h\}) = \{g\}$, then $C(\{f, g, h\}) = \{f\}$ by WARNI (both g and h are globally blocked), thus f globally blocks h, thus $C(\{f, h\}) = \{f\}$. The following claims are then easily shown by mimicking the preceding proof (whose notation is freely used): Utility acts constitute equivalence classes; one

[5] Taking revealed preference *really* seriously, one might remark that in many applications including statistical decision theory, agents can randomize, thus one cannot observe choice from finite menus but only from the corresponding mixture sets, which here coincide with their convex hulls. The directly revealed preference might then be even more incomplete than "true" preference, namely if $f \succsim g$ yet choice from the convex hull $co\{f, g\}$ is a proper mixture of the two. In general, this leads to substantial complications (Stoye (2011b) and references therein). These do not arise here because if $\lambda f + (1-\lambda)g \in C(co\{f, g\})$, then $\lambda f + (1-\lambda)g \in C(co\{\lambda f + (1-\lambda)g, g\})$ by the property α implication of WARNI, thus $f \in C(co\{f, g\})$ by independence.

can proceed as if the range of U were unbounded; and $f \succ g \Leftrightarrow f \ominus g \succ p_0$. Consider now any act f that takes at least three different utility values, then $\underline{f}_{\underline{E}}\overline{f} \succ f \succ \overline{f}_{\overline{E}}\underline{f}$ by strict admissibility. Transitivity then yields $\overline{f}_{\overline{E}}\underline{f} \succ p_0 \Rightarrow f \succ p_0$ and $f \succ p_0 \Rightarrow \underline{f}_{\underline{E}}\overline{f} \succ p_0$, while symmetry yields $\underline{f}_{\underline{E}}\overline{f} \succ p_0 \Leftrightarrow \overline{f}_{\underline{E}}\underline{f} \succ p_0 \Leftrightarrow \overline{f}_{\overline{E}}\underline{f} \succ p_0$. Hence, attention can be restricted to acts of form $p_E q$ as before. The last part of the previous proof can then be mimicked to show that $\min_{s \in \mathcal{S}} u \circ f(s) \geq 0$ is necessary for $f \succ p_0$; on the other hand, $\min_{s \in \mathcal{S}} u \circ f(s) \geq 0$ and $f \neq p_0$ is sufficient by axiom C.2. ■

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