Axioms for Minimax Regret Choice Correspondences

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Abstract

This paper unifies and extends the recent axiomatic literature on minimax regret. It compares several models of minimax regret, shows how to characterize the according choice correspondences in a unified setting, extends one of them to choice from convex (through randomization) sets, and connects them by defining a behavioral notion of perceived ambiguity. Substantively, a main idea is to behaviorally identify ambiguity with failures of independence of irrelevant alternatives. Regarding proof technique, the core contribution is to uncover a dualism between choice correspondences and preferences in an environment where this dualism is not obvious. This insight can be used to generate results by importing findings from the existing literature on preference orderings.

Keywords: Minimax regret, ambiguity, ambiguity aversion, multiple priors, choice correspondences.

JEL classification codes: C44, D81.

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1 Introduction

1.1 Motivation

The minimax regret decision criterion was suggested in Savage's [50] reading of Wald [65] and has since seen occasional use in statistics. Interest in minimax regret has recently increased among econometricians and economic theorists alike.[1] Its decision theoretic foundations, the classic reference for which is Milnor [44], were revisited as well, specifically in recent work by Hayashi [31] and Stoye [61].[2]

This paper provides further insight into axiomatic characterizations of minimax regret. It is partly motivated by the observation that, although all of the aforementioned authors talk about something they call "minimax regret," there are significant differences between their conception of this object. I hope to clarify some discussions of regret by elaborating on these differences. Building on this, I provide a unified characterization of different "minimax regret" objects in a common framework and axiomatic system. I also give a behavioral characterization of perceived ambiguity and extend the axiomatic characterization of one such model to convex sets, i.e. agents who can randomize. These are this paper's main substantive contributions. One leitmotif that connects them is the idea to identify perceived ambiguity with violations of independence of irrelevant alternatives (IIA). As IIA is the one standard axiom that will *not* be imposed in this paper, this identification resembles Ghirardato, Macheroni, and Marinacci's [23] identification of perceived ambiguity with violations of von Neumann-Morgenstern independence. This similarity will turn out to be much more than semantic. The paper's main technical contribution is a proof technique which recovers a dualism between choice correspondences and preference orderings in an environment where this dualism is not obvious. This allows one to adapt existing results on preferences to statements about regret-based choice correspondences. An exception to this is the extension to convex sets, which requires substantial work beyond recovering said dualism.

I unify previous axiomatizations of regret along two dimensions. First, one can think of a minimax regret preference ordering or of the according choice correspondence. All applications are phrased in terms of the preference ordering, and it is this ordering that Stoye [61] analyzes. However, this ordering is menu dependent, i.e. the ranking of acts can depend on the feasible set within which they are compared. As a result, axiomatizations of minimax regret preferences are at tension with the revealed

[1] The strictly first appearance of minimax regret seems to be Niehans [45]. Examples for use in statistics are DasGupta and Studden [14], Droge [16] [17], and Eldar, Ben-Tal, and Nemirovski [18]. See Stoye [59] for a compilation of relevant references in econometrics. Uses in economic theory include Bergemann and Schlag [5] [6], Eozenou, Rivas, and Schlag [20], Halpern and Pass [29], Hayashi [32], Linhart and Radner [37], Renou and Schlag [47] [48], and Schlag [51]; substantively, some of these papers overlap with econometrics.

[2] Both also do other things to which this paper is less related – Stoye [61] by looking at more preference orderings, Hayashi [31] by formalizing a notion of smooth (non-minimax) regret aversion.

preference paradigm. To illustrate, consider the statement $f \succ g \succ h$, where the menu in question is $\{f, g, h\}$. Choice from the menu reveals that $f \succ g$ and $f \succ h$, but not that $g \succ h$. If preferences do not depend on menus, an obvious dualism between preferences and choice correspondences resolves the problem because $g \succ h$ is revealed by choice from $\{g, h\}$. With menu-dependent preferences, this dualism breaks down, and choice from $\{g, h\}$ will not reveal the ranking of those same acts within $\{f, g, h\}$. Indeed, the second part of $f \succ g \succ h$ need not map onto revealed preference for g over h in *any* menu. One could, therefore, prefer to restrict attention to the relevant choice correspondence, an approach introduced to the literature on regret by Hayashi [31]. The present paper adopts the choice correspondence approach, but on a deeper level, it shows that the two perspectives continue to be tightly related because a somewhat different dualism can be uncovered.

Second, minimax regret can be thought of as presuming no priors, endogenous priors, or exogenous priors.

(i) No priors: Milnor [44] and Stoye [61] axiomatize the preference ordering represented by (the negative of)

$$\max_{s \in \mathcal{S}} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\},$$

where f and g are acts in a menu M, $\mathcal{S}$ is a state space, and u is an expected utility functional; I will explain notation in detail below. The idea here is that $\mathcal{S}$ reflects the objective ambiguity inherent in a situation.

In applications, prior-less minimax regret was recently used by Bergemann and Schlag [5], Manski [40] [41] [43], Schlag [51] [52], and Stoye [57] [62]. For example, Manski considers the problem of treatment choice – be it assignment to on-the-job training programs or to medical treatment – as a statistical decision problem and compares the risk functions generated by different statistical treatment rules. He advocates the use of minimax regret risk as decision criterion, but certainly not the use of priors.[3] Indeed, as this version of minimax regret is the only one that can be interpreted without any notion of priors, it is the one that frequentist statisticians must have in mind and that corresponds to Savage's [50] original suggestion. It is also used in Linhart and Radner [37] and most of Halpern and Pass [29].

[3] Notation in statistics is typically somewhat different. Statisticians postulate a set of conceivable data generating processes and a loss function. A risk function maps any combination of true data generating process and statistical decision rule onto the implied expectation of loss. Noting that data generating processes correspond to states of the world, statistical decision rules to acts, and loss to (the negative of) utility, risk functions are seen to map onto the functional u (i.e., utility acts). The parallels are explained in detail in Stoye [58].

(ii) Endogenous Priors: Consider also minimization of

$$\max_{\pi \in \Gamma} \int \left(\max_{g \in M} u \circ g(s) - u \circ f(s) \right) d\pi,$$

where the "set of priors" $\Gamma \subseteq \Delta \mathcal{S}$ is behavioral or "as if." Mathematically, this generalizes the previous expression because Γ could equal $\Delta \mathcal{S}$; note also that it simplifies to expected utility if Γ is a singleton. This approach is in line with Gilboa and Schmeidler [26] and the large literature that builds on them. It may be the most interesting one for a theorist since we cannot typically observe people's (sets of) beliefs; hence, these axiomatizations are most revealing regarding a theory's observable implications. Endogenous prior minimax regret was axiomatized by Hayashi [31] subject to the caveat that Γ must intersect the interior of $\Delta \mathcal{S}$; it was recently applied to single-agent decision problems by Hayashi [32] [33] and to games by Renou and Schlag [47] [48] and Halpern and Pass [29].

(iii) Exogenous Priors: An intermediate possibility is that the representation is as in (ii) but Γ is a feature of the environment. The quintessential example is the robust (multi-prior) Bayesian literature (Berger [7]), which first specifies a set of priors and hence the extent of ambiguity in a decision situation and subsequently thinks about how to make decisions. Indeed, many contributions make the first and not the second step (e.g., Wasserman and Kadane [66]), and the literature "does not as yet contain substantial work on how exactly a specific action should be chosen" (Zen and DasGupta [67]; see also Arias, Hernández, Martín, and Suárez [2]). If specification of set valued beliefs precedes the contemplation of action, they should arguably be considered part of the decision theoretic environment, and beliefs revealed by choices should be axiomatically linked to them. In particular, a characterization of minimax regret with exogenous priors amounts to an axiomatic foundation for Γ-minimax regret (Berger [7]; see Bergemann and Schlag [5] or Chamberlain [10] for applications to economics). The symbol Γ for sets of priors is chosen to emphasize this link.

To repeat, which of these possibilities appears most interesting depends on the desired application. The typical, namely descriptive and behavioral, application in economic theory will rely on case (ii). On the other hand, although this paper is rooted in the revealed preference paradigm, it is partly motivated by statistical and econometric applications. A frequentist application avoids priors and consequently, is an example of case (i). When statistical applications do use (sets of) priors, these priors should typically be thought of as exogenous; these cases therefore fall under (iii).[4]

[4] Case (iii) also ties in with a recent literature that treats (possibly vague) prior information as part of the decision theoretic environment. See Gajdos, Tallon, and Vergnaud [22] for an early reference and Giraud and Tallon [27] for an advocacy and further references.

1.2 Overview and Brief Summary

This paper provides characterizations of minimax regret choice correspondences for all these cases and connects them via a behavioral characterization of perceived ambiguity. For endogenous prior minimax regret, the result resembles a finding in Hayashi (2008, theorem 4); in this case, the main contribution lies in the embedding of results, the removal of the aforementioned restriction of Γ, and a change of perspective that will be explained now. More generally, the paper contains the following innovations.

- Axioms are designed to think about regret as a novel theory of ambiguity aversion, where such aversion is injected into an otherwise standard setting by relaxing independence of irrelevant alternatives (respectively the weak axiom of revealed preference) and not von Neumann-Morgenstern independence. To emphasize this symmetry, notation is very similar to Gilboa and Schmeidler [26] and other references. The idea, which will be reiterated when relevant axioms are discussed, is that the comparison of two options is revealed ambiguous if revealed preference between them can switch as choice problems change.

- Technically, it is observed that, while a failure of independence of irrelevant alternatives apparently disconnects choice correspondences from preference orderings, the axioms are still strong enough to recover a tight link between the two. This link is the focus of lemma 2 below and will be explained in detail later.

- The scope of the literature on regret is expanded by considering randomized decision making. This extension should be of special interest for statistical decision theory because statisticians can and do randomize. From a revealed preference perspective, randomization makes the problem harder by convexifying all choice sets and thereby reducing the domain on which axioms can be asserted. The problem is resolved, and results are recovered, for prior-less minimax regret, though it is open for endogenous prior minimax regret.

An overview of the paper's structure goes as follows. Section 2 describes the decision theoretic environment and states axioms. It then provides a lemma that generates the aforementioned connection between choice correspondences and preference orderings. Characterizations of prior-less minimax regret (by importing a result of Stoye [61] that builds on Milnor [44]) and endogenous prior minimax regret (by importing Gilboa and Schmeidler [26]) follow easily. One can also use a link to previous work on multi-prior Pareto criteria by Bewley [8] and Ghirardato, Macheroni, and Marinacci [23] to characterize a notion of ambiguity perception. This informs an axiomatization that identifies the object Γ with an exogenous set of priors. The section concludes by recovering the characterization of prior-less minimax regret if agents have access to objective randomization devices. Section 3 offers a conclusion and comparisons to other notions of regret in the literature.

2 Axiomatic Analysis

2.1 Preliminaries and Axioms

The setup is inspired by Anscombe and Aumann [1]. There is a set $\mathcal{S}$ of states of the world s, endowed with an algebra Σ of events; a set $\mathcal{X}$ of outcomes x; and a set $\mathcal{F}$ of acts $f, g, \ldots$. The only restrictions on $\mathcal{S}$, $\mathcal{X}$, and Σ are that $\mathcal{X}$ must have at least two elements and that theorems 3 and 7 require the existence of three distinct, nonempty events. An act f is a Σ-measurable, finite step function $f : \mathcal{S} \to \Delta\mathcal{X}$ that maps states s onto finite outcome distributions $f(s)$. An act is *constant* if f does not depend on s. With the usual abuse of notation, $\Delta\mathcal{X}$ is embedded in $\mathcal{F}$ by using $p, q, \ldots$ to denote both lotteries in $\Delta\mathcal{X}$ and the corresponding constant acts. Mixtures of acts are identified with statewise mixtures, i.e. $h \equiv \lambda f + (1-\lambda)g$ is characterized by $h(s) = \lambda f(s) + (1-\lambda)g(s)$. The word "convex" henceforth denotes closure under such mixture. The decision maker can choose from a finite, nonempty menu $M \subseteq \mathcal{F}$. For any menu M, $\lambda M + (1-\lambda)g$ denotes the menu generated by replacing every $f \in M$ with the analog mixture: $\lambda M + (1-\lambda)g \equiv \{f' \in \mathcal{F} : f' = \lambda f + (1-\lambda)g, f \in M\}$. I do not presume existence of a preference ordering but of a choice correspondence C that maps every M onto some nonempty $C(M) \subseteq M$. (This is a good moment to emphasize that in this paper, $\subseteq$ and $\subset$ are distinct symbols.) I also define the problem of choosing an act after state s has been learned. The according choice correspondence will be labelled C_s. As is standard in the literature, I impose some notion of dynamic consistency by assuming that choice after revelation of s is equivalent to choice from constant acts; more formally, $f \in C_s(M)$ iff $f(s) \in C(\{g(s) : g \in M\})$. I call an act f *strictly potentially optimal* in M if there exists s s.t. $C_s(M) = \{f\}$. Finally, for future use, let $\Delta\mathcal{S}$ denote the set of finitely additive probability measures on $(\mathcal{S}, \Sigma)$.

The following axioms on C will be maintained throughout.

Axiom 1 ***Nontriviality***

$$\exists M : C(M) \subset M.$$

Axiom 2 ***Monotonicity***

If $f \in M$, $g \in C(M)$, and $f \in C_s(\{f, g\})$ for all s, then $f \in C(M)$.

Axiom 3 ***Independence***

$$C(\lambda M + (1-\lambda)f) = \lambda C(M) + (1-\lambda)f.$$

Axiom 4 ***Independence of Irrelevant Alternatives (IIA) for Constant Acts***

Let M and N consist of constant acts, then

$$C(M \cup N) \cap M \in \{C(M), \varnothing\}.$$

Axiom 5 ***Independence of Never Strictly Optimal Alternatives (INA)***

Let M and N be such that $C_s(M \cup N) \cap M \neq \varnothing$ for all s. Then

$$C(M \cup N) \cap M \in \{C(M), \varnothing\}.$$

Axiom 6 ***Mixture Continuity***

Fix any menu M and acts $f \notin M$ s.t. $C(M \cup \{f\}) = \{f\}$, $g \in M$, and $h \in \mathcal{F}$. Then there exists $\lambda \in (0,1)$ s.t. $C(M \cup \{\lambda f + (1-\lambda)h\}) = \{\lambda f + (1-\lambda)h\}$ and $\lambda g + (1-\lambda)h \notin C(M \cup \{f, \lambda g + (1-\lambda)h\})$.

Axiom 7 ***Ambiguity Aversion***

$C(M)$ is the intersection of M with a convex set. That is, fix any acts f, g, scalar $\lambda \in [0,1]$, and menu $M \supseteq \{f, g, \lambda f + (1-\lambda)g\}$, then $\{f, g\} \subseteq C(M) \Rightarrow \lambda f + (1-\lambda)g \in C(M)$.

Some remarks on the axioms are in order. Monotonicity states that if the agent would choose f from $\{f, g\}$ in every state of the world, then she cannot revealed prefer g over f in any menu. It is the revealed preference equivalent of the axiom of the same name in Gilboa and Schmeidler [26] and other references. Independence requires that choice is invariant under mixing of *entire menus* with some act. One intuition for this comes from the following thought experiment. Suppose an agent chooses from a menu, but then learns that her choice will be actualized only conditional on heads in a previous coin toss; she has no control over what will happen conditional on tails. Then it can be argued that her choice behavior should not be affected.[5] The same adaptation of independence was recently used by Eliaz and Ok [19] and Ortoleva [46].

Axioms 5 and 6 touch upon a crucial, and controversial, feature of minimax regret, namely that it violates independence of irrelevant alternatives (IIA).[6] This axiom would translate into the present setting as

$$C(M \cup N) \cap M \in \{C(M), \varnothing\}, \forall M, N,$$

[5] The leap from the thought experiment to the axiom relies on a hidden assumption of compound lottery reduction or "reversal of order." Here as in many other references, that axiom is implicit in the notation for mixture acts; see Seo [55] for a treatment that makes it explicit.

[6] I here refer to the IIA exiom for individual decision making (not social choice) due to Arrow [3, definition C.4]. This axiom is equivalent to Sen's [54] properties α and β, which translate into present notation as

$$f \in C(M \cup N) \cap M \Longrightarrow f \in C(M),$$
$$\{f, g\} \in C(M) \Longrightarrow C(M \cup N) \cap \{f, g\} \in \{\{f, g\}, \varnothing\}.$$

the former of which goes back at least to Chernoff [11].

meaning that preferences revealed by $C(M \cup N)$ cannot contradict those revealed by $C(M)$: Choice from $M \cup N$ is either strictly revealed preferred to $C(M)$, in which case it cannot contain elements of the latter, or it is revealed indifferent to $C(M)$, in which case it contains all of it.

In this paper, IIA will only be imposed on restricted domains. Specifically, axioms 5 and 6, as well as several axioms to come, introduce a leitmotif: Comparison of $C(M)$ and $C(M \cup N)$ may reveal violations of IIA if the ambiguity perceived in a choice problem changes as one expands M to $M \cup N$. This cannot be the case if both M and N consist of constant acts, because then there is no ambiguity to begin with; hence axiom 5. Furthermore, axiom 6 (INA) specifies that this cannot happen if acts added to menus are not strictly potentially optimal. An intuition for this is that the agent's attitude to one and the same outcome in different states may be influenced by what could have been achieved in a state, hence the nature of an act's ambiguity may change with this information. This intuition is obviously related to the concept of regret, and INA accordingly plays a major role in enforcing regret-based choices.[7]

Ambiguity aversion translates an axiom proposed by Schmeidler [53]; see Milnor [44] for a precursor. Note that unlike in most other contexts, ambiguity aversion is not a weakening of independence because it would follow from the latter only in conjunction with IIA.

Consider now the following axiom.

Axiom 8 ***Symmetry***

Fix any menu M and disjoint events $E_1, E_2 \in \Sigma \backslash \{\varnothing\}$ s.t. any act $f \in M$ is constant on E_1 as well as E_2. Define f' by

$$f'(s) = \begin{cases} f(s)\|_{s \in E_2}, & s \in E_1 \\ f(s)\|_{s \in E_1}, & s \in E_2 \\ f(s) & \textit{otherwise} \end{cases}.$$

Let the function $(\cdot)' : 2^{\mathcal{F}} \to 2^{\mathcal{F}}$ map every set of acts $N \subseteq \mathcal{F}$ onto $N' \equiv \{f' : f \in N\}$. Then

$$C(M') = (C(M))'.$$

In words, symmetry states the following. Take any two events such that all acts in a menu are constant on either event, then exchanging the consequences of the events throughout the menu does not affect choices. This is certainly not an innocuous condition – the two events might be of very different size with respect to some measure on the state space. Indeed, symmetry enforces a strong attitude of ignorance regarding elements of $\mathcal{S}$, specifically a refusal to weight them according to some importance criterion like subjective probability. The idea that prior ignorance about events should be modelled in

[7] Krähmer and Stone [35] give a similar, informational motivation for regret, although their technical notion of regret is outside this paper's scope. Essentially the same axiom as INA is used in Hayashi [31], but the original source is Milnor [44, "special row adjunction"].

this way is due to Arrow and Hurwicz [4], who specify similar axioms for choice correspondences; see also Cohen and Jaffray [13] for a preference-based formulation that is otherwise analog to the above.

Symmetry is implausible if one has available, and wishes to consider, prior information about states, respectively if one wishes to model agents who have and use such information. On the other hand, if no prior information exists, the axiom is compelling because decisions would otherwise be sensitive to manipulations of $\mathcal{S}$, e.g. the relabeling of states or their duplication via conditioning on trivial events (Arrow and Hurwicz [4]). These observations are as they should be, given that symmetry will turn out to characterize prior-less minimax regret. For the intermediate case of *vague* prior information – not enough to commit to a prior but sufficient to cast doubt on symmetry –, I would point to the characterization of exogenous prior minimax regret below.

I finally consider the following axiom. Say that a menu has a *state independent outcome distributions* if the set $\{p \in \Delta X : \exists f \in M, f(s) = p\}$ does not vary with s. In words, the set of feasible outcome lotteries is constant across states. Any menu consisting of constant acts induces state independent outcome distributions, but a menu can have this property without containing any constant act. Now consider the following.

Axiom 9 ***C-Betweenness When Outcome Distributions are State Independent***

For any act f, constant act p, scalar $\lambda \in (0,1)$, and menu $M \supseteq \{p, f, \lambda f + (1-\lambda)p\}$ with state independent outcome distributions, if $p \notin C(M)$ and $f \notin C(M)$, then $\lambda f + (1-\lambda)p \notin C(M)$.

C-betweenness for state independent outcome distributions is related to betweenness (Chew [12], Dekel [15]), which states that if two acts are ranked indifferent, then they are also ranked indifferent to any mixture between them. The possibility that the two acts are chosen over the mixture is already excluded by ambiguity aversion; the new axiom ensures that the mixture cannot be chosen over the indifferent acts.[8]

The motivation of c-betweenness is related to the usual motivation of c-independence: It limits the scope of ambiguity aversion, that is, of preference for mixtures. Intuitively, a decision maker might strictly prefer mixtures because they constitute a hedging of bets across ambiguous states. C-independence states that mixture with a constant act cannot constitute a hedging of bets. Axiom 9 further tightens the conditions under which a hedge is denied, thus weakening the axiom. The first tightening is that the menu must have state independent outcome distributions. The idea here is to once again acknowledge that ambiguity can arise not just because outcomes differ across states, but also because the evaluation of outcomes might depend on what could have been achieved. The

[8] The axiom's formulation is somewhat stronger than directly backed by the intuition because a notion of revealed indifference between f and p is not readily available and, therefore, not presumed. However, I would argue that intuitively, the case where f and p are equally is surely the critical one for the axiom's plausibility.

restriction shuts down the latter channel: Observed outcomes may still be informative about which state occurred, and hence about which act should have been chosen, but if the set of ex post feasible outcome distributions is state independent, then none of this information matters for what could have been achieved. The second tightening is that the axiom only applies to mixture of f with constant acts. Substantively, this weakening sharpens the focus on preferences for or against randomization. By applying for all constant acts from very bad to very good ones, c-independence additionally limits the dependence of ambiguity attitudes upon stakes.

I conclude this section by stating a straightforward result: A subset of axioms that will be maintained throughout implies that C extends some expected utility choice correspondence $\widetilde{C}$ in the sense of agreeing with it on choice from constant acts.

Lemma 1 *C fulfils axioms 3, 4, and 6 iff choices from sets of constant acts are rationalized by von Neumann-Morgenstern expected utility, i.e. there exists a unique (up to affine transformation) function $U : \mathcal{X} \mapsto \mathbb{R}$ s.t. the restriction $\widetilde{C}$ of C to menus $\widetilde{M}$ consisting of constant acts p is*

$$\widetilde{C}\left(\widetilde{M}\right) = \arg\max_{p \in \widetilde{M}} \int U(x)dp.$$

2.2 Characterizations of Minimax Regret

This section is devoted to characterizing different versions of minimax regret. The key to this is contained in the following lemma.

Lemma 2 *C fulfils axioms 1 through 6 iff it can be represented as*

$$C(M) = \arg\min_{f \in M} I(r \circ (f, M)),$$

where the finite step function $r \circ (f, M) : \mathcal{S} \mapsto \mathbb{R}^+$ is defined by

$$\begin{aligned} (r \circ (f, M))(s) &\equiv \max_{g \in M} u \circ g(s) - u \circ f(s) \\ u \circ f(s) &\equiv \int U(x)df(s) \end{aligned}$$

with U as in lemma 1 and furthermore nonconstant, and the functional I, which takes values in $\mathbb{R}^+$, is mixture continuous ($\{\lambda : I(\lambda r + (1-\lambda)r') \geq I(r'')\}$ and $\{\lambda : I(\lambda r + (1-\lambda)r') \leq I(r'')\}$ are closed for all r, r', r''), monotonic in two different senses ($r \geq r'$ for all s implies $I(r) \geq I(r')$; $r > r'$ for all s implies $I(r) > I(r')$), and homothetic.

Lemma 2 is modelled on lemma 3.3 in Gilboa and Schmeidler [26], with notational differences indicating substantive ones. It tightly limits the ways in which ambiguity can affect choices – they must reveal a preference ordering, here represented by value functional I, over objects $r \circ (f, M)$ that one might call *regret acts*.

The proof of lemma 2 contains four crucial steps, the second and third of which are among this paper's main insights.

- Lemma 1, monotonicity, and INA jointly imply that acts can be identified with *utility acts* $u \circ f : \mathcal{S} \mapsto \mathbb{R}$. This idea is standard except for the observation that it requires merely INA instead of IIA.

- Independence can be used to restrict attention to the set $\mathcal{M}_0$ of menus whose join or ex-post utility frontier, $\{\max_{g \in M} u \circ g(s)\}_{s \in \mathcal{S}}$, is everywhere zero. This insight was anticipated more than half a century ago (Chernoff [11, theorem 2]) but seems to have gone unused since; e.g., it is missing in Milnor [44] and Borodin and El-Yaniv [9]. An intuition is that if there exists an act h with $u \circ h(s) = -\max_{g \in M} u \circ g(s)$, then independence implies

$$f \in C(M) \Longleftrightarrow \frac{1}{2}f + \frac{1}{2}h \in C\left(\frac{1}{2}M + \frac{1}{2}h\right),$$

  but by construction, $\max_{g \in \frac{1}{2}M + \frac{1}{2}h} u \circ g(s) = 0$ for every s. Thus, C is determined by its restriction to $\mathcal{M}_0$. (The formal proof involves some detours because existence of h is not guaranteed.)

- INA implies that one can construct a menu-independent preference ordering $\succsim_C$ which rationalizes the restriction of C to $\mathcal{M}_0$. Specifically, for all acts f, g with nonpositive utility range, define

$$f \succsim_C g \Longleftrightarrow \exists M \in \mathcal{M}_0 : f \in C(M), g \in M,$$

  that is, $f \succsim_C g$ if f is sometimes chosen in the presence of g. Then $\succsim_C$ generates the restriction of C to $\mathcal{M}_0$ as choice correspondence. In "nice" cases, $\succsim_C$ is furthermore a complete ordering that can be represented by a value functional I with the asserted properties. (Note in particular that independence does not imply that $\succsim_C$ is independent, however it yields that $\succsim_C$ is homothetic.)

- There are two reasons why $\succsim_C$ may be incomplete. First, if the range of U is bounded, acts whose utility is very low may not be chosen from any menu in $\mathcal{M}_0$, thus there is no notion of revealed preference among them. At this point (though not later), the problem is straightforwardly resolved by observing that any extension of $\succsim_C$ will induce $\succsim_C$ as choice correspondence and that homotheticity of $\succsim_C$ informs an extension which leaves all other properties of $\succsim_C$ intact. Second, $\succsim_C$ may fail to order *any* acts with strictly negative utility range, a salient example being a Bayesian decision maker whose prior is concentrated on one state. In this case, the agent's behavior only distinguishes a set of "choosable" acts from a set of "non-choosable" ones; however, there still exists a criterion function I with the claimed properties that rationalizes C on $\mathcal{M}_0$.

Returning to the substantive importance of lemma 2, its main point is as follows. The proofs in Gilboa and Schmeidler [26], as well as many related papers, initially establish that preferences can be represented by a value functional I operated on utility acts. Lemma 2 is analogous to this, except that I is operated on regret acts which absorb any menu dependence of the agent's evaluation of acts. Substantively, this leads to a separation of risk aversion as well as menu dependence of preferences, both of which are absorbed by $r \circ (f, M)$, and attitude to uncertainty about s, which is reflected in I.[9] Formally, it re-instates a dualism between choice correspondences and preferences, albeit a more intricate one than is generated by IIA. The practical benefit of this dualism is that existing axiomatic results for preferences can be imported if $\succsim_C$ can be shown to fulfil their if-sides.

This idea will now be exploited to generate different axiomatizations of choice correspondences. The first results are theorems 3 and 4, which characterize minimax regret choice correspondences with no respectively endogenous priors.

Theorem 3 ***Prior-less Minimax Regret***

Let Σ contain at least three distinct events. A choice correspondence fulfils axioms 1 through 8 iff it can be represented as

$$C(M) = \arg\min_{f \in M} \max_{s \in \mathcal{S}} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\}$$

with u as in lemma 2.

Given lemma 2, theorem 3 is established by applying Stoye [61, theorem 1(iii)] to $\succsim_C$.

Theorem 4 ***Endogenous Prior Minimax Regret***

A choice correspondence fulfils axioms 1 through 7 and 9 iff it can be represented as

$$C(M) = \arg\min_{f \in M} \max_{\pi \in \Gamma} \int \left(\max_{g \in M} u \circ g(s) - u \circ f(s) \right) d\pi$$

for some compact, convex $\Gamma \subseteq \Delta\mathcal{S}$. Here, Γ is unique and u is as in lemma 2.

To establish this result, one needs to show that $\succsim_C$ is c-independent, after which Gilboa and Schmeidler [26] can be invoked. This argument has two main ingredients. First, axiom 9 together with ambiguity aversion ensures that $\succsim_C$ fulfils a preference version of c-betweenness: $f \sim_C p$ implies that $f \sim_C \lambda f + (1 - \lambda) p$ for all $\lambda \in (0, 1)$. This ensures that indifference sets of $\succsim_C$ are collections of

[9] This separation is less clear in Gilboa and Schmeidler [26, lemma 3.3] because there, the preferences encoded by I may change if u is replaced with a positive affine transformation of itself; since this is considered an equivalence transformation, I is not really identified separately from u. To achieve this separation, one also has to impose c-independence of I (Ghirardato, Maccheroni, and Marinacci [24]). The problem does not arise here: A positive affine transformation of u induces a positive *linear* transformation of $r \circ (f, M)$; since $\succsim_C$ is homothetic, the choice correspondence cannot be affected.

rays emanating from constant acts. Homotheticity of $\succsim_C$ prevents these rays from fanning out or in, leading to c-independence. Thus, the "stake independence" aspect of c-independence is delivered by independence of C, which drives homotheticity of $\succsim_C$.

Theorem 4 resembles a finding in Hayashi [31, theorem 2]. The settings are slightly different, and, while there is much overlap between axioms, c-betweenness here and "constant-regret independence of regret premium" there are rather dissimilar. Also, monotonicity displaces Hayashi's admissibility axiom. As a result of this weakening, Γ need not intersect the interior of $\Delta\mathcal{S}$; for example, the case of a Bayesian who assigns zero probability to some states is not excluded. The main contribution of theorem 4, however, lies not in the statement of the result but in its derivation (specifically, exhibiting a tight link to Gilboa and Schmeidler [26]) and interpretation (specifically, embedding it in other results and deriving regret as a model of ambiguity aversion). While the representation in theorem 3 is obviously a special case of the one in theorem 4, it is also worth noting that theorem 3 does not use c-betweenness. Thus, while prior-less minimax regret can be enforced by invoking theorem 4 and then observing that the maximal set of priors is the only one consistent with symmetry, the resulting characterization would not be tight. This resembles the observation (Stoye [61]) that for preferences, c-independence is not needed to characterize prior-less maximin utility.

2.3 A Characterization of Perceived Ambiguity

Lemma 2 can also be used to develop a behavioral notion of perceived ambiguity in the framework of theorem 4. The construction requires one preliminary step:

Definition 1 *For any choice correspondence C, define $\trianglerighteq_C$ by $f \trianglerighteq_C g$ iff*

$$[\lambda g + (1-\lambda)p \in C(M) \Rightarrow \lambda f + (1-\lambda)p \in C(M)],$$
$$\forall \lambda \in (0,1), p \text{ constant, } M \supseteq \{\lambda f + (1-\lambda)p, \lambda g + (1-\lambda)p\}.$$

Remark 1 *If the range of U is unbounded and theorem 4 applies, one can equivalently define $f \trianglerighteq_C g$ iff*

$$[g \in C(M) \Rightarrow f \in C(M)], \forall M \supseteq \{f, g\}.$$

To get an intuition, the reader should inspect the simplified definition: $f \trianglerighteq_C g$ if there exists *no* menu in which g is chosen over f. This "revealed unambiguous preference" is in general transitive but incomplete. I relate it to ambiguity to once again emphasize the conceptual link between menu dependence and ambiguity. The idea is that if $f \trianglerighteq_C g$, then the comparison between f and g is context independent, hence revealed unambiguous. The mixing of both acts with a constant act is a technical detail that becomes necessary if U is bounded. It enables one to manipulate the utility range of acts while only rescaling (by a state-independent factor) their utility differences across states. Given that independence is already presumed, this would appear rather innocuous.[10]

Comparative ambiguity perception can be characterized as follows.

Definition 2 *The choice correspondence C reveals (weakly) more perceived ambiguity than C' if*

$$f \trianglerighteq_C g \Longrightarrow f \trianglerighteq_{C'} g.$$

Theorem 5 ***Characterization of Comparative Ambiguity Aversion***

Assume that theorem 4 applies. The choice correspondence C reveals (weakly) more perceived ambiguity than C' iff both can be represented by the same utility function U in conjunction with sets of priors Γ (to represent C) and Γ' (to represent C') s.t. $\Gamma \supseteq \Gamma'$.

The theorem lends further support to the identification of revealed unambiguous preference with ambiguity perception because it tightly links perceived ambiguity to sets of priors, which prima facie represent the decision maker's perception of ambiguity in her environment. It should be kept in mind, however, that this paper is behavioral and hence, Γ is not claimed to map onto any real objects, including true sets of beliefs. One might therefore want to more cautiously interpret the result as a characterization of $\trianglerighteq_C$ that illustrates conceptual consistency of this paper's terminology. (See also the similar discussion in Ghirardato, Macheroni, and Marinacci [23].)

To understand why the theorem is true, let the range of U be unbounded and recall that the link between C and $\succsim_C$ from lemma 2 is partly established by mapping arbitrary menus into $\mathcal{M}_0$ via mixing with a "normalizing" act. This link can be used to show that $f \trianglerighteq_C g$ iff $\lambda f + (1-\lambda)h \succsim_C \lambda g + (1-\lambda)h$ for any probability $\lambda \in [0, 1)$ and any act h. But this means that $f \trianglerighteq_C g$ is the independence-abiding, incomplete preference ordering defined by Ghirardato, Macheroni, and Marinacci's [23, see also [8]] to capture perceived ambiguity. Theorem 5 then follows by importing their results. It implies that comparative ambiguity perception coincides with comparative regret aversion in Hayashi [31]. Once the link between $\succsim_C$ and $\trianglerighteq_C$ has been established, this is expected because Hayashi's notion of comparative regret aversion adapts Ghirardato and Marinacchi's [25] notion of comparative ambiguity aversion, which coincides with the one of Ghirardato, Macheroni, and Marinacci [23] for maxmin utility preferences – but $\succsim$ is just such a preference.

[10] It may even be surprising that the feature is needed. To see why, let there be just two states, i.e. $\mathcal{S} = \{s_1, s_2\}$, identify acts f with utility vectors $(u \circ f(s_1), u \circ f(s_2)) \in \mathbb{R}^2$, and assume that the choice correspondence is the priorless one. Then $f = (1, 0)$ and $g = (-1, 1)$ should not be $\trianglerighteq_C$-comparable; indeed, g is chosen over f from $\{f, g, (-2, 3)\}$. But if U is bounded above by 1, then f incurs regret of exactly 1 but g incurs regret of at least 2 in every menu containing both, hence the simplified defionition of $\trianglerighteq_C$ would imply $f \trianglerighteq_C g$. Mixing f and g with constant acts resolves the problem: $C(\{(1/3, 0), (-1/3, 1/3), (-2/3, 1)\}) = \{(-1/3, 1/3)\}$.

An interesting application of theorem 5 is axiomatization of minimax regret where the set of priors is exogenous. Let there exist a compact, convex object $\Gamma^* \subseteq \Delta\mathcal{S}$ and consider linking it axiomatically to the object Γ in the preceding representation. One motivation for this approach is statistical decision theory, in particular Γ-minimax regret; see Giraud and Tallon [27] for the argument that models with exogenous imprecise information should receive more attention in theory more generally. The necessary axioms are as follows.[11]

Axiom 10 Γ^****-Monotonicity***

$$\int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^* \Longrightarrow f \trianglerighteq_C g.$$

Axiom 11 Γ^****-Ambiguity***

$$f \trianglerighteq_C g \Longrightarrow \int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^*.$$

To understand Γ^*-monotonicity, recall that in view of lemma 1, the standard monotonicity axiom can be slightly rewritten: $f \in C_s(\{f, g\}), \forall s$ is equivalent to $u \circ f(s) \geq u \circ g(s), \forall s$. This reveals that Γ^*-monotonicity is intuitively similar to monotonicity, but strengthens it. It imposes that if the comparison of f and g is commonsensically unambiguous to anybody who accepts utility function U and priors Γ^*, then revealed preferences should indeed not reveal any ambiguity in the sense of menu dependence, and should furthermore have the obvious direction. The same axiom is used, to a similar effect as here, by Gajdos, Tallon, and Vergnaud [22].

Conversely, Γ^*-ambiguity stipulates that revealed preference is unambiguous in the sense of menu-independent only if the according comparison of acts is commonsensically unambiguous given Γ^*. Equivalently, if the comparison of two acts under Γ^* is ambiguous in the sense that either act is favored by some element of Γ^*, then the choice correspondence reflects this ambiguity in the sense of violating IIA. This relates Γ^*-ambiguity to the aforementioned leitmotif, namely that violations of IIA should be driven by ambiguity.

In short, Γ^*-monotonicity can be thought of as ensuring that the decision maker does not see more ambiguity than is encoded in Γ^*; Γ^*-ambiguity can be thought of as ensuring that she does not see less. The axioms' effects accord with these intuitions.

Corollary 6 ***Exogenous Priors Minimax Regret***

Assume that theorem 4 applies. Then:

[11] I find it most intuitive to state the axioms in terms of u, but of course, existence of this object depends on previous axioms. To avoid this, one can rephrase $\int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^*$ as $\int f(s) d\pi \in C\left\{\int f(s) d\pi, \int g(s) d\pi\right\}, \forall \pi \in \Gamma^*$. The statements are equivalent by lemma 1.

(i) *A choice correspondence satisfies* Γ^**-monotonicity iff* $\Gamma \subseteq \Gamma^*$*.*

(ii) *A choice correspondence satisfies* Γ^**-ambiguity iff* $\Gamma \supseteq \Gamma^*$*.*

Corollary 6 is true because axioms 10 and 11 restrict C to reveal less (respectively more) ambiguity than the minimax regret ordering with the same utility function and set of priors Γ^*. It builds on theorem 4, but also connects theorem 3 to theorem 4 because it identifies prior-less minimax regret with maximal perceived ambiguity. This yields a yet alternative (and tight) characterization of prior-less minimax regret and again illustrates conceptual consistency of terminology. After all, we would surely think of the absence of any prior information as maximizing ambiguity.[12]

2.4 Prior-less Minimax Regret when Agents can Randomize

The above and all other existing results on minimax regret assume that agents cannot randomize, thus allowing for axioms to be asserted for choice from nonconvex sets.[13] (Indeed, given finiteness of menus, every nondegenerate menu is also nonconvex.) But statisticians can and do randomize, and minimax regret statistical decision rules frequently prescribe randomization (Manski [41] [43], Schlag [52], Stoye [57] [62]). What's more, the ambiguity aversion axiom maintained throughout enforces a weak preference for randomization. It appears less than satisfying to assume that agents (weakly) want to randomize yet to prevent them from doing so even though the agents in questions (i.e., statisticians) clearly can.

Thus, assume now that agents can choose from convex hulls of finite menus.[14] This will be captured by a change in notation: In this section only, a menu M consists of a finite set of acts and all possible mixtures over those, thus it is the convex hull of the corresponding object from earlier sections. This convexification severely affects even a minimal sense of revealed preference between two acts. Choice from $\{f,g\}$ at least reveals what can be thought of as preference between them in menu $\{f,g\}$. Choice from the convex hull of $\{f,g\}$ – heneforth denoted $co\{f,g\}$ – might (and, for ambiguity averse decision makers, frequently will) contain only proper mixtures of the two. This renders revealed preferences

[12] As an aside, by identifying $\trianglerighteq_C$ with the incomplete, independent relation at the heart of Ghirardato, Macheroni, and Marinacci [23], one can easily re-import axioms 10 and 11 and corollary 6 into their setting to yield a characterization of exogenous prior maximin utility.

[13] I thank a referee for raising the question answered in this section. The question was raised, and I would be interested in the analysis, for all results in this paper, but the other extensions are beyond this paper's scope.

[14] The notation implies that randomization by the decision maker and objective risk modelled by outcome lotteries are compounded; in other words, lotteries over sets of acts equal convex hulls of those sets. Of course, this is subject to the caveat from footnote 5. Also, while an experimantal protocol in which one observes randomized choice is conceivable, in observational data one will typically only recover choice frequencies, i.e. an imperfect measure of choice probabilities. I leave deeper analysis of these issues for further research, noting in passing that if models of deterministic choice are taken to actual data, measurement error typically becomes an issue as well.

inherently incomplete. What's more, even an unrestricted independence of irrelevant alternatives assumption only corresponds to the weak but not the strong axiom of revealed preference; as soon as it is restricted to convex sets, it fails to imply the latter and, therefore, transitivity of revealed preference.[15] An additional loss of structure occurs with respect to continuity, which will be adapted as follows:

Axiom 6* Mixture Continuity for Convex Hulls

Fix any menu M, act $f \in \mathcal{F}$ s.t. $C(co\,(M \cup \{f\})) \cap M = \varnothing$, $g \in M$, and $h \in \mathcal{F}$. Then there exists $\lambda \in (0,1)$ s.t. $g \notin C(co(M \cup \{\lambda f + (1-\lambda)h\}))$ and $\lambda g + (1-\lambda)h \notin C(co\,(M \cup \{f, \lambda g + (1-\lambda)h\}))$.

On the if-side of this axiom, it is not required that $C(co\,(M \cup \{f\})) = \{f\}$, so the axiom may feel somewhat stronger. This is attenuated by a slight weakening: The axiom cannot apply if $f \in M$. More importantly, the then-side became correspondingly weaker: The requirement that $C(M \cup \{\lambda f + (1-\lambda)h\}) = \{\lambda f + (1-\lambda)h\}$ is weakened to $g \notin C(co(M \cup \{\lambda f + (1-\lambda)h\}))$. Indeed, $\lambda f + (1-\lambda)h \notin C(co(M \cup \{\lambda f + (1-\lambda)h\}))$ is a relevant possibility.[16] Regarding other axioms, note also that the conclusion of INA now is $C(co(M \cup N)) \cap M \in \{C(M), \varnothing\}$ and that ambiguity aversion now simply requires C to be convex-valued.

As a result of these complications, I am not able to recover lemma 2 nor a direct analog to theorem 4. However, it is possible to recover the conclusion of theorem 3 with substantial additional effort and, interestingly, by using c-betweenness.

Theorem 7 ***Prior-less Minimax Regret When Agents can Randomize***

Consider a setting exactly as in the preceding theorems, but where agents can randomize, thus menus M are convex hulls of finiute menus. Let Σ contain at least three distinct events. Then a choice correspondence fulfils axioms 1 through 5, 6, and 7 through 9 iff it can be represented as*

$$C(M) = \arg\min_{f \in M} \max_{s \in \mathcal{S}} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\}$$

with u as in lemma 2.

While the result looks familiar, much has changed below the surface. In particular, the revealed preference ordering $\succsim_C$ generated from choice problems in $\mathcal{M}_0$ is now highly incomplete. It is still true, however, that every completion $\succsim$ of $\succsim_C$ will induce C as choice correspondence on $\mathcal{M}_0$, and

[15] See Stoye [60] for a detailed analysis of revelation of menu-independent preferences by choice from convex sets.

[16] Consider a three-state world in which acts are identified with utility vectors $(u_1, u_2, u_3) \in \mathbb{R}^3$. If $\succsim_C$ is rationalized by prior-less minimax regret, then $(0,0,0) \succ_C (0,-1,-1)$ and $(0,-1,-1) \succ_C (-2,-1,-1)$; to see the latter, consider choice from $co(\{(0,-1,-1),(-2,-1,-1),(-5,0,-5),(-5,-5,0)\})$. Yet $(-2\lambda,-\lambda,-\lambda) \succ_C (0,-1,-1)$ for *no* $\lambda \in (0,1)$ because a randomization that puts weight $(1/(1+\lambda))$ on $(-2\lambda,-\lambda,-\lambda)$ fares better than either act.

theorem 3 can be recovered by specifying an appropriate completion $\succsim$ of $\succsim_C$. The trick is to assign to every act the value of a suitably defined certainty equivalent, specifically the best constant act that is not strictly $\succsim_C$-preferred to it. (This somewhat intricate definition is necessary because revealed $\succsim_C$-indifference between an act and its certainty equivalent cannot be presumed.[17]) An argument driven by symmetry and ambiguity aversion then shows that $\succsim$ is the prior-less maximin utility ranking. With this finding in place, one can show that $\succsim$ extends $\succsim_C$. In an interesting twist, this last step seems to require c-betweenness to ensure that acts and their certainty equivalents are either $\succsim_C$-indifferent or $\succsim_C$-noncomparable but never strictly $\succsim_C$-ordered; this is why c-betweenness appears in the axioms here and not in theorem 3. Although substantial portions of the proof of lemma 2 are invoked in the proof of theorem 7 and intuitions are certainly related, the result therefore stands as the only main result in this paper that is not really a corollary of lemma 2.[18]

3 Conclusion

This paper unified some of the recent, axiomatic literature on minimax regret. It adopted choice correspondences as general framework, but demonstrated that even without independence of irrelevant alternatives, there exists a tight link between axiomatizations of preference orderings and of choice correspondences. Under restrictions shared by many regret-based approaches, results of the latter kind can be generated from existing results of the former kind. I used this insight to provide a number of minimax regret characterizations, namely of prior-less, of endogenous multiple prior, and of exogenous multiple prior minimax regret, a characterization of perceived ambiguity, and the extension of prior-less regret to choice from convex (through randomization) sets.

The framework is intended to be rather universal and covers all of the applications cited in the introduction. Nonetheless, it is impossible to unify in one paper every concept that has been labeled "regret." I therefore conclude by clarifying the relation between minimax regret as formalized here and some other notions in economics and related fields.

Statistical decision theory is one motivation of this paper, and many but not all uses of regret there coincide with formalisms here. To comply with this paper's notion of minimax regret, statistical decision rules or estimators must be compared to an "oracle estimator" which is best among the

[17] In the setting of footnote 11, $(-1,-1)$ is the certainty equivalent of $(0,-1)$, yet the two are not revealed indifferent: Any $M \in \mathcal{M}_0$ that contains $\{(0,-1),(-1,-1)\}$ also contains an act $(x,0)$, and the agent will pick some mixture of $(0,-1)$ and $(x,0)$ over either of $\{(0,-1),(-1,-1)\}$.

[18] The wording is hedged because I do not have an example for necessity of c-betweenness, so this question is open. The original parts of the argument are related to, and borrow from, concurrent work reported in Stoye [60]. The present exercise is harder, however, because the restriction to menus in $\mathcal{M}_0$ means that acts need not be $\succsim_C$-comparable to their certainty equivalents.

feasible ones, given hypothetical knowledge of the true state of the world. Examples include the treatment choice problems in Manski [40] [42] [43], Schlag [52], and Stoye [56] [57] [62], where the oracle treatment rules are no-data rules that respond to true expectations, but also the estimation problems in Droge [16] [17], Eldar, Ben-Tal, and Nemirovski [18], and Hansen [30], where they are the ex post best from certain classes of estimators. An incompatible example, however, is the "predictive entropy regret" approach of Sweeting, Datta, and Ghosh [64], which benchmarks against a specific act, thereby is not menu dependent, and in the terminology of (economists') decision theory, rather looks like maximin utility with a specific, state dependent utility function.

The word "regret" resounds in everyday language, and some readers may accordingly be interested in it from the vantage point of psychology or behavioral economics. From that perspective, one may critically remark that regret here benchmarks against an "omniscient" ex post stage in which the true state of the world has been revealed. This does not correspond to a situation that the decision maker anticipates to ever experience, so one may hesitate to identify this paper's notion of regret with anticipated feelings. If the latter are a core motivation, one may want to explore regret preferences that benchmark against what the decision maker will, in fact, learn from outcome realizations. This is the motivation of Krähmer and Stone [35]. Interestingly, it can lead to preferences against information: Of two otherwise identical acts, the one whose outcomes are less correlated with, and hence less informative about, other acts' potential outcomes may be strictly preferred. The approach has not, to my knowledge, been axiomatized.

A well known invocation of regret in economic theory is due to Loomes and Sugden [38, see also Fishburn [21], Sugden [63]]. This approach has in common with the current one that regret is evaluated from an omniscient view; some papers partially justify this by imposing independence of outcome lotteries across acts, thus removing one wedge between the "realistic ex post" and the omniscient information stage. Major differences to the present perspective are the imposition of a concave transformation of regret and the use of subjective priors rather than max min-operators. Indeed, the dichotomy of risk versus ambiguity/Knightian uncertainty is not emphasized in this approach, and the relevant papers switch between imposing objective probabilities (Loomes and Sugden [38]) and a Savage environment (Sugden [63]).

Finally, Sarver [49] proposed a model of regret that embeds it in the recent literature on menu dependent preferences. One difference to the present approach is that Sarver's utility function combines conventional utility with an additive regret penalty. More importantly, he follows Kreps [36] and Gul and Pesendorfer [28] in axiomatizing preferences over menus; in the language of utility maximization, the axiomatization is of the value functional. For these authors' motivation, this is an ingenious device. For the present paper's notion of regret, its use would be less obvious because the minimax regret value functional has an unusual interpretation. If choice problem M causes more minimax regret than problem N, this may mean that learning is more valuable in M, and this intuition may well inform a future axiomatization – but it does not imply that N is more desirable by any commonsensical standard. Indeed, it could easily be the case that any option in M dominates any element of N in utility terms. An instructive axiomatization of this value functional would, accordingly, have to be quite different from the present contribution.

A Proofs

Lemma 1 Define $p \succsim q \Leftrightarrow p \in C(\{p, q\})$ for constant acts, then $\succsim$ is a complete and transitive preference that rationalizes C (Arrow [3]) and is easily verified to fulfil Herstein and Milnor's [34] axioms, hence the result.

Preliminaries to all subsequent results. Recall that lemma 1 applies. For any act f, define the mapping ("utility act") $u \circ f : \mathcal{S} \mapsto \mathbb{R}$ by $u \circ f(s) \equiv \int U(x)df(s)$ and use the notation $\geqq$ $[\gg]$ as follows: $f \geqq [\gg] g \Leftrightarrow u \circ f(s) \geq [>] u \circ g(s), \forall s$. Observe that in the statement of monotonicity, $[f \in C_s(\{f, g\}), \forall s]$ can now be written as $f \geqq g$. Also use the following shorthand: For any number v in the (convex hull of the) range of U, p_v is the constant act with corresponding utility value.

No information is lost by identifying every act f with $u \circ f$. To see this, fix any menus M and M' such that there exists a one-to-one mapping $(.)' : M \to M'$ with $u \circ f' = u \circ f$ for every $f \in M$. Then $f \in C(M) \Leftrightarrow f' \in C(M')$. To see this, consider $C(M \cup M')$. By INA, $C(M \cup M') \cap M \in \{C(M), \varnothing\}$ and $C(M \cup M') \cap M' \in \{C(M'), \varnothing\}$. Nonemptiness of C and monotonicity now jointly imply that $C(M \cup M') = C(M) \cup C(M')$, and monotonicity (applied to f and f') then implies the claim. With abuse of notation, I therefore identify acts with utility acts, that is, I conflate f and $u \circ f$.

Nonconstancy of U is necessary for lemma 1, monotonicity, and nontriviality to be mutually consistent. It implies that after normalization, $U^{-1}(-1)$ and $U^{-1}(1)$ can be assumed to exist. Hence, any finite, Σ-measurable step function $u : \mathcal{S} \to [-1, 1]$ can be identified with a feasible act f. This specifically includes p_0. Independence implies that $C(\lambda M + (1-\lambda)p_0) = \lambda C(M) + (1-\lambda)p_0 = \lambda C(M)$. Hence, C is homogeneous of degree one: For any menu M and scalar $\lambda \in (0, 1)$, the menu λM exists and $C(\lambda M) = \lambda C(M)$.

Lemma 2 In this and the next two proofs, I only show "only if."

Step 1: Defining a revealed preference relation over "regret acts." For any menu M, let $\overline{f}_M$ denote the act with $u \circ \overline{f}_M(s) = \max_{f \in M} u \circ f(s)$. By finiteness of acts and menus, this "oracle act" or join always exists. Let $\mathcal{M}_0$ denote the set of menus M s.t. $\overline{f}_M = p_0$, i.e. menus whose ex post best possible utility is zero in every state. Let $\mathcal{F}_-$ denote the set of acts with nonpositive utility range. Define the relation $\succsim_C$ on $\mathcal{F}_- \times \mathcal{F}_-$ as follows:

$$f \succ_C g \Longleftrightarrow \exists M \in \mathcal{M}_0 : f \in C(M), g \in M \backslash C(M),$$
$$f \sim_C g \Longleftrightarrow \exists M \in \mathcal{M}_0 : f \in C(M), g \in C(M),$$

i.e. $\succsim_C$ is preference directly revealed from choice problems in $\mathcal{M}_0$. INA easily implies asymmetry of $\succ_C$ and disjointness of $\succ_C$ and $\sim_C$. To see transitivity, let $f \succsim_C g \succsim_C h$ and let M be the menu in which $g \succsim_C h$ is revealed, then INA and $f \succsim_C g$ imply $f \in C(M \cup \{f\}) \Rightarrow f \succsim_C h$.

To gather some more properties, assume that U is not bounded from below; the case where this fails will be handled in step 3. Homogeneity of degree 1 of C then implies homotheticity of $\succsim_C$: $f \succsim_C g \Leftrightarrow [\lambda f \succsim_C \lambda g, \forall \lambda > 0]$. Now assume also that there exist $p \ll p_0$ and $M \in \mathcal{M}_0$ s.t. $p \in C(M)$; the case where this fails will be handled in step 4. Then $\succsim_C$ is complete: Fix any acts f and g, let p and M be the act and menu whose existence was just assumed, and let $\lambda > 0$ be s.t. $f \geqq \lambda p$, then $C(\lambda M \cup \{f, g\}) \cap \{f, g\} \neq \varnothing$ by INA and monotonicity. Using completeness, $f \geqq g \Rightarrow f \succsim_C g$ follows easily from monotonicity, and closedness of $\{\lambda : \lambda f + (1-\lambda) g \succsim_C h\}$ and $\{\lambda : \lambda f + (1-\lambda) g \precsim_C h\}$ follows easily from mixture continuity. To see that $f \gg g \Rightarrow f \succ_C g$, fix f and g with $f \gg g$, then $f \succsim_C g$ was shown. Suppose by contradiction that $f \sim_C g$, then homotheticity and monotonicity yield that all acts $h \ll p_0$ are mutually indifferent, in particular $g \sim_C p_\lambda$ for any $\lambda < 0$. At the same time, $C(\{p_0, p_{-1}\}) = \{p_0\}$ by lemma 1 and the normalization of U, hence $p_0 \succ_C g$; but both facts together contradict mixture continuity upon adding p_0 to the menu in which $f \sim_C g$ is revealed and then mixing p_0 with p_{-1}.

Every act f is $\succsim_C$-indifferent to exactly one constant act p, henceforth also called its certainty equivalent. To see that there exists at least one such act, let the constant acts p, q be s.t. $p \geqq f \geqq q$, thus $p \succsim_C f \succsim_C q$, then existence of λ^* s.t. $\lambda^* p + (1-\lambda^*) q \sim_C f$ follows from completeness and mixture continuity. That there is at most one certainty equivalent follows from $p \gg q \Rightarrow p \succ_C q$ and transitivity. Hence, $\succsim_C$ can be represented by a real-valued function J that maps f onto the utility value of its certainty equivalent.

Step 2: Characterizing C in terms of revealed preference over regret acts. The restriction of C to $\mathcal{M}_0$ equals the choice correspondence induced by $\succsim_C$:

$$C(M) = \{f \in M : g \in M \Rightarrow f \succsim_C g\} = \arg\max_{f \in M} J(u \circ f)$$

for all $M \in \mathcal{M}_0$. To see this, fix $M \in \mathcal{M}_0$, then $f \in C(M) \Rightarrow f \succsim_C g$ for all $g \in M$, but also $f \in M \backslash C(M) \Rightarrow g \succ_C f$ for some $g \in M$. To extend the representation to arbitrary menus, fix any M and let $\lambda = 1/\max_{f \in M, s \in \mathcal{S}} |u \circ f(s)|$. Then λM (and, by implication, $-\overline{f}_{\lambda M}$) has utility range in

$[-1,1]$ and, therefore, exists. Use $C(\lambda M) = \lambda C(M)$ and independence to write

$$\frac{\lambda}{2}C(M) + \frac{1}{2}\left(-\overline{f}_{\lambda M}\right) = \frac{1}{2}C\left(\lambda M\right) + \frac{1}{2}\left(-\overline{f}_{\lambda M}\right) = C\left(\frac{\lambda}{2}M + \frac{1}{2}\left(-\overline{f}_{\lambda M}\right)\right)$$
$$\Longrightarrow C(M) = \frac{2}{\lambda}C\left(\frac{\lambda}{2}M + \frac{1}{2}\left(-\overline{f}_{\lambda M}\right)\right) + \frac{1}{\lambda}\overline{f}_{\lambda M},$$

but $\frac{\lambda}{2}M + \frac{1}{2}\left(-\overline{f}_{\lambda M}\right) \in \mathcal{M}_0$, hence $C\left(\frac{\lambda}{2}M + \frac{1}{2}\left(-\overline{f}_{\lambda M}\right)\right) = \arg\max_{f \in \frac{\lambda}{2}M + \frac{1}{2}\left(-\overline{f}_{\lambda M}\right)} J(u \circ f)$. It follows that

$$C(M) = \frac{2}{\lambda}\arg\max_{f \in \frac{\lambda}{2}M + \frac{1}{2}\left(-\overline{f}_{\lambda M}\right)} J(u \circ f) + \frac{1}{\lambda}\overline{f}_{\lambda M} = \arg\min_{f \in M} I(r \circ (f, M)),$$

where $I = -J$ and the last step also used homotheticity of $\succsim_C$. By step 1, I has all the properties claimed in the lemma.

Step 3: Taking care of bounded utility. Assume now that U is bounded from below, but continue to assume that $p \in C(M)$ for some $p \ll p_0$ and $M \in \mathcal{M}_0$; then it is w.l.o.g. to let $p = p_{-1}$ (possibly by rescaling U). Monotonicity then implies that $\succsim_C$ is complete on the domain of utility acts with range in $[-1, 0]$ and, on this domain, has all the properties collected in step 1.

The crucial observation is that any ordering $\succsim$ on $\mathcal{F}_- \times \mathcal{F}_-$ which extends $\succsim_C$ induces C as choice correspondence on $\mathcal{M}_0$, that is, $\{f \in M : g \in M \Rightarrow f \succsim g\} = \{f \in M : g \in M \Rightarrow f \succsim_C g\}$ for any $M \in \mathcal{M}_0$. To see this, let $f \in C(M)$, then $f \succsim_C g$, thus $f \succsim g$, for any $g \in M$. Let $f \in M \backslash C(M)$, then $g \succ_C f$, thus $g \succ f$, for some $g \in M$.

The appropriate extension is the homothetic one: For any $f, g \in \mathcal{F}_-$, $f \succsim g$ iff $\rho f \succsim_C \rho g$, where $\rho = 1/\max_{s \in \mathcal{S}} \max\{|u \circ f(s)|, |u \circ g(s)|\}$. This is well-defined (and complete) because $\succsim_C$ orders ρf and ρg. It extends $\succsim_C$ because $\succsim_C$ is homothetic (up to completeness). Now step 2 goes through as before.

Step 4: Taking care of a special case. Returning to the last paragraph of step 1 (and bearing in mind that asymmetry of $\succ_C$, disjointness of $\succ_C$ and $\sim_C$, and transitivity of $\succsim_C$ were already shown), continue to assume that U is unbounded from below but assume now that no constant act $p \ll p_0$ is chosen from any menu $M \in \mathcal{M}_0$. Monotonicity and INA then imply that no act $f \ll p_0$ is chosen from any menu $M \in \mathcal{M}_0$. Consider now any act $h \neq p_0$ that fails $h \ll p_0$, i.e. it achieves zero utility in some state, and some constant act $p \leqq h$. Then monotonicity, applied to $\{p_0, h\}$, implies $p_0 \succsim_C h$, while the fact that p is chosen from no menu $M \in \mathcal{M}_0$ implies that either $h \succ_C p$ or h and p are noncomparable. Suppose $p_0 \succ_C h \succ_C p$, then mixture continuity, applied to p_0 and the menu in which $h \succ_C p$ is revealed, yields that $\lambda p \succ_C h$ for some $\lambda > 0$, a contradiction (recall no constant act is chosen from any $M \in \mathcal{M}_0$). Thus, $p_0 \succ_C h$ implies that h and p are noncomparable. Now suppose $p_0 \sim_C h$, then $h \succ_C p$ follows from INA upon adding p to the menu in which $p_0 \sim_C h$ is revealed. It follows that

$\succsim_C$ can be represented by a set $I_0 \subset \mathcal{F}_-$ s.t. $C(M) = M \cap I_0$ for all $M \in \mathcal{M}_0$. Let $f \in I_0$ and let $E = \{s \in S : u \circ f(s) = 0\}$, then homotheticity and monotonicity imply that $p_{0E}g \in I_0$ for arbitrary $g \in \mathcal{F}_-$. Thus, I_0 can be characterized by a set of events $\Sigma_0 \subseteq \Sigma$ s.t. $I_0 = \{p_{0E}f : E \in \Sigma_0, f \in \mathcal{F}_-\}$, and $\succsim_C$ can be characterized by identifying J with sup-distance (in utility terms) from I_0, fulfilling all the properties claimed.[19] Step 2 goes through as before, and adaptation for the case of bounded (from below) U is as in step 3.

Theorem 3 In view of step 3 in the preceding proof, assume that U is unbounded from below (or that the $\succsim_C$ used in the following incorporates a homothetic extension). By the properties collected in step 1 of lemma 2, $\succsim_C$ then fulfils all axioms used in Stoye [61, theorem 1(iii)] except for ambiguity aversion and symmetry. Ambiguity aversion ($f \sim_C g \Rightarrow \lambda f + (1-\lambda)g \succsim_C f$ for all $\lambda \in [0,1]$) follows easily from INA and ambiguity aversion of C. It remains to derive symmetry ($f \succsim_C g \Leftrightarrow f' \succsim_C g'$, where $(.)'$ is as in the text). Thus, fix acts $f, g \in \mathcal{F}_-$ and events $E_1, E_2 \in \Sigma$ s.t. f and g are constant on E_1 and E_2. Define $E = E_1 \cup E_2$ and $\rho = \max_{s \in \mathcal{S}} \max\{|u \circ f(s)|, |u \circ g(s)|\}$. If $E \subset \mathcal{S}$, consider $M = \{p_{0E}p_{-2}, p_{-1}, p_{-2E}p_0\}$, then symmetry and ambiguity aversion jointly imply $p_{-1} \in C(M)$. Consider now $N = M \cup \{f/\rho, g/\rho\}$. As p_{-1} is dominated by both f/ρ and g/ρ, INA and monotonicity jointly imply $C(N) \cap \{f/\rho, g/\rho\} \neq \varnothing$, hence $f \succsim_C g \Leftrightarrow f/\rho \in C(N)$. If $E = \mathcal{S}$, the same argument applies but starting from $M = \{p_{0E_1}p_{-2}, p_{-1}, p_{-2E_1}p_0\}$. In either case, symmetry of $\succsim_C$ is implied upon comparing N and N', the menu generated from N by interchanging the consequences of E_1 and E_2.

Thus $\succsim_C$ is priorless maximin utility: $f \succsim_C g$ iff $\min_{s \in \mathcal{S}} u \circ f(s) \geq \min_{s \in \mathcal{S}} u \circ g(s)$. Substituting into lemma 2 yields

$$C(M) = \arg\min_{f \in M} \max_{s \in \mathcal{S}} \left\{ \max_{g \in M} u \circ g(s) - u \circ f(s) \right\}$$

as required. Individual necessity of most axioms is easy. For necessity of three events, let $\mathcal{S} = \{s_1, s_2\}$ and let $\succsim_C$ be represented by $u \circ f(s_1) + u \circ f(s_2)$.

Theorem 4 Let U be unbounded from below; the extension to bounded U is as before. Suppose that no act $p \ll p_0$ is chosen from any menu $M \in \mathcal{M}_0$, thus the choice correspondence is the one discovered in step 4 of the proof of lemma 2. Consider any $E \in \Sigma/\{\varnothing, \mathcal{S}\}$, then I_0 contains $C(\{p_{0E}p_{-2}, p_{-2E}p_0\})$ and therefore one of those two acts, but it cannot contain both because ambiguity aversion would then imply $p_{-1} \in I_0$. Hence, either $E \in \Sigma_0$ or $\mathcal{S} \backslash E \in \Sigma_0$ but not both. At the same time, monotonicity implies that Σ_0 is closed under the formation of supersets. Now let $\pi^* = 1\{E \in \Sigma_0\}$, then $\pi^* \in \Delta\mathcal{S}$.

[19] This possibility may appear exotic, but one instance of it ($f \sim_C p_0$ iff $\max_{s \in \mathcal{S}} u \circ f(s) = 0$) rationalizes the "minimin regret" choice correspondence that collects all potential best responses, and another one (there exists $s^* \in \mathcal{S}$ s.t. $f \sim_C p_0$ iff $u \circ f(s^*) = 0$) corresponds to Bayesianism with degenerate prior.

The decision maker can be characterized as maximizing $\int u \circ f(s) d\pi^*$, fulfilling the theorem.[20]

Now assume that some $p \ll p_0$ is chosen from some menu $M \in \mathcal{M}_0$, thus $\succsim_C$ has the properties collected in step 1 of lemma 2. As $\succsim_C$ also transparently inherits ambiguity aversion, it remains to show c-independence: $f \succsim_C g \Leftrightarrow \lambda f + (1-\lambda)p \succsim_C \lambda g + (1-\lambda)p$.

Axiom 9 implies that there exists no menu $M \in \mathcal{M}_0$ s.t. $f, p \in M \backslash C(M)$ and $\lambda f + (1-\lambda)p \in C(M)$ for some (f, p, λ). To see this, assume that M exists. As acts and menus are finite, there exists a finite partition $\Sigma_M \subset \Sigma$ of $\mathcal{S}$ s.t. every act $f \in M$ is constant on every event $E \in \Sigma_M$. Define $\mathcal{U}_M = \{v \in \mathbb{R}^- : u \circ f(s) = v \text{ for some } f \in M, s \in \mathcal{S}\}$ and let $M^* = M \cup \{p_{vE} p_{\min \mathcal{U}_M} : v \in \mathcal{U}_M, E \in \Sigma_M\}$. As every element of M^* is dominated by some element of M, INA and monotonicity jointly imply that $C(M \cup M^*) \cap M = C(M)$. But M^* has state independent outcome distributions, thus axiom 9 is contradicted.

Next, $\succsim_C$ fulfils what might be called c-betweenness for preferences: $f \sim_C p \Leftrightarrow f \sim_C \lambda f + (1-\lambda)p$ for all $\lambda \in (0,1)$. To see "$\Rightarrow$," assume that $f \sim_C p$ and let M be the menu in which this is revealed. Then $\lambda f + (1-\lambda)p \in C(M \cup \{\lambda f + (1-\lambda)p\})$ by INA and ambiguity aversion, thus $f \in C(M \cup \{\lambda f + (1-\lambda)p\})$ by INA and the preceding paragraph's conclusion, thus $f \sim_C \lambda f + (1-\lambda)p$. Now suppose that $f \sim_C \lambda f + (1-\lambda)p$ for *some* $\lambda \in (0,1)$. This implies $f \sim_C p$, establishing "$\Leftarrow$." To see this, recall that f has a (unique) certainty equivalent q, hence $f \sim_C \lambda f + (1-\lambda)q$ by "$\Rightarrow$," hence $\lambda f + (1-\lambda)p \sim_C \lambda f + (1-\lambda)q$ by transitivity, hence $p = q$ (because $f \gg g \Rightarrow f \succ_C g$).

Next, $f \sim_C g \Leftrightarrow \lambda f + (1-\lambda)p \sim_C \lambda g + (1-\lambda)p$. For this and the following step, assume that $f, g \ll p_0$; the preceding paragraph's result can be used to extend indifference sets to boundary acts. For $p = p_0$, the claim is immediate from homotheticity of $\succsim_C$. Else, suppose $f \sim_C g$, then $f \sim_C g \sim_C \gamma p$ for a unique $\gamma > 0$. By the preceding paragraph's result and transitivity, $\rho f + (1-\rho)\gamma p \sim_C \rho g + (1-\rho)\gamma p$ for any $\rho \in (0,1)$. Let $\delta = (1 - \lambda + \gamma\lambda)/\gamma$ and $\rho = \gamma\lambda/(1-\lambda+\gamma\lambda)$, then $\lambda f + (1-\lambda)p = \delta[\rho f + (1-\rho)\gamma p]$ and $\lambda g + (1-\lambda)p = \delta[\rho g + (1-\rho)\gamma p]$. Homotheticity of $\succsim_C$ now yields $\lambda f + (1-\lambda)p \sim_C \lambda g + (1-\lambda)p$. The converse follows from the reverse argument, using the "if"-direction of c-betweenness.

Finally, $f \prec_C g \Leftrightarrow \lambda f + (1-\lambda)p \prec_C \lambda g + (1-\lambda)p$. Suppose $f \prec_C g$, then by monotonicity etc., there exists (a unique) $\gamma \in [0,1)$ s.t. $\gamma f \sim_C g$. The preceding paragraph's conclusion implies that $\lambda\gamma f + (1-\lambda)p \sim_C \lambda g + (1-\lambda)p$ for all constant acts p, which in turn implies $\lambda f + (1-\lambda)p \prec_C \lambda g + (1-\lambda)p$. If $f \succ_C g$, use the same argument with the roles of f and g reversed. Suppose $\lambda f + (1-\lambda)p \prec_C \lambda g + (1-\lambda)p$, then $f \sim_C g$ would violate the preceding paragraph's conclusion, and $f \succ_C g$ would violate this paragraph's preceding conclusion, hence $f \prec_C g$.

Individual necessity of most axioms is easy. For necessity of c-betweenness, let $\succsim_C$ be represented by $-\left[\int (u \circ f(s))^2 d\pi\right]^{1/2}$, where $\pi \in \Delta\mathcal{S}$.

[20] The intuitive example is that π^* is concentrated on one state s^*, but to see that permitting *finitely* additive measures is crucial here (and for the theorem), let $\mathcal{S} = [0,1]$ with Borel-algebra and consider $\Sigma_0 = \{E : (0, \varepsilon) \subseteq E \text{ for some } \varepsilon > 0\}$.

Remark 1 Straightforward from independence.

Theorem 5 Recall that theorem 4 applies. If Γ is a singleton, the decision maker is Bayesian and the theorem is trivially true. Else, there exist distinct events $E, F \in \Sigma$ that have positive probability under some $\pi \in \Gamma$. It follows that $\lambda p_{-1} \in C(\{p_{-1E}p_0, p_{-1F}p_0, \lambda p_{-1}\}$ for some $\lambda > 0$; also using monotonicity, $\succsim_C$ is complete on the domain of utility acts with range in $[-\lambda, 0]$. For the remainder of this proof, rescale U s.t. $\lambda = 1$. Let $\mathcal{F}_{\mathbb{R}}$ collect all utility acts f, g with arbitrary (including positive) range, and define the relation $\succsim$ on $\mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{\mathbb{R}}$ as follows: $f \succsim g$ iff $\min_{\pi\in\Gamma} \int u \circ f(s)d\pi \geq \min_{\pi\in\Gamma} \int u \circ g(s)d\pi$, with Γ the same object that characterizes $\succsim_C$. In words, $\succsim$ is the obvious extension of $\succsim_C$ to the larger domain. Define the incomplete relation $\trianglerighteq$ by

$$f \trianglerighteq g \Longleftrightarrow \lambda f + (1-\lambda)h \succsim \lambda g + (1-\lambda)h, \forall \lambda \in (0,1], h \in \mathcal{F}_{\mathbb{R}}.$$

Then by Ghirardato, Macheroni, and Marinacci [23, proposition 5], there exists a unique, compact, convex, set of probabilities $\widetilde{\Gamma}$ s.t.

$$f \trianglerighteq g \Longleftrightarrow \int u \circ f(s)d\pi \geq \int u \circ g(s)d\pi, \forall \pi \in \widetilde{\Gamma}.$$

Now, [23, proposition 19, used with $\alpha = 1$] implies that $\widetilde{\Gamma} = \Gamma$. The theorem then follows from [23, theorem 6] upon observing that $\trianglerighteq_C = \trianglerighteq$. The latter holds because the following two statements are equivalent:

$$\exists \lambda \in (0,1], h \in \mathcal{F}_{\mathbb{R}} : \lambda f + (1-\lambda)h \prec \lambda g + (1-\lambda)h,$$

$$\exists \lambda, p, M \supseteq \{\lambda f + (1-\lambda)p, \lambda g + (1-\lambda)p\} : C(M) \cap \{\lambda f + (1-\lambda)p, \lambda g + (1-\lambda)p\} = \{\lambda g + (1-\lambda)p\}.$$

To see equivalence, assume $\lambda f + (1-\lambda)h \prec \lambda g + (1-\lambda)h$, then $\rho(\lambda f + (1-\lambda)h) + (1-\rho)p_{-1/2} \prec \rho(\lambda g + (1-\lambda)h) + (1-\rho)p_{-1/2}$ for any $\rho \in (0,1)$ by c-independence of $\succsim$. Choose ρ small enough s.t. both $\rho(\lambda f + (1-\lambda)h) + (1-\rho)p_{-1/2}$ and $\rho(\lambda g + (1-\lambda)h) + (1-\rho)p_{-1/2}$ have utility range in $[-1,0]$ and also $\rho(1-\lambda)h$ has utility range in $[-1,1]$. As $\succsim$ extends $\succsim_C$, $\rho(\lambda g + (1-\lambda)h) + (1-\rho)p_{-1/2}$ is chosen over $\rho(\lambda f + (1-\lambda)h) + (1-\rho)p_{-1/2}$ in some menu $M \in \mathcal{M}_0$. By independence, $\frac{\rho\lambda}{2}g + \frac{1-\rho}{2}p_{-1/2}$ is then chosen over $\frac{\rho\lambda}{2}f + \frac{1-\rho}{2}p_{-1/2}$ in $\frac{1}{2}M + \frac{1}{2}(-\rho(1-\lambda))h$. (To precisely replicate the desired conclusion, make the identification $\lambda \mapsto \frac{\rho\lambda}{2}$, $p \mapsto p_{-(1-\rho)/(4-2\rho\gamma)}$, and $M \mapsto \frac{1}{2}M + \frac{1}{2}(-\rho(1-\lambda))h$.)

Conversely, assume that $\lambda g + (1-\lambda)p$ is chosen over $\lambda f + (1-\lambda)p$ in some menu M. Let ρ be small enough s.t. ρM has utility range in $[-1,1]$, then independence yields

$$\frac{\rho\lambda}{2}f + \frac{\rho(1-\lambda)}{2}p - \frac{\rho}{2}\overline{f}_M \prec_C \frac{\rho\lambda}{2}g + \frac{\rho(1-\lambda)}{2}p - \frac{\rho}{2}\overline{f}_M,$$

thus $\lambda f + (1-\lambda)p - \overline{f}_M \prec \lambda g + (1-\lambda)p - \overline{f}_M$ by homotheticity of $\succsim_C$ and the fact that $\succsim$ extends $\succsim_C$, thus the conclusion (with $h = p - \overline{f}_M/(1-\lambda)$).

Corollary 6 Define $\trianglerighteq^*$ by

$$f \trianglerighteq^* g \Longleftrightarrow \int u \circ f(s) d\pi \geq \int u \circ g(s) d\pi, \forall \pi \in \Gamma^*.$$

Then axiom 10 states that $f \trianglerighteq^* g \Rightarrow f \trianglerighteq g$, whereas axiom 11 states that $f \trianglerighteq g \Rightarrow f \trianglerighteq^* g$. The claim follows from theorem 5.

Theorem 7

Adapting the preliminaries. Lemma 1 continues to hold. This is shown in Stoye [60], the argument is repeated here for completeness. For this paragraph only, define the relation $\trianglerighteq_C$ on constant acts by $p \trianglerighteq_C q$ iff $p \in C(co\{p,q\})$. Then $\trianglerighteq_C$ is complete: Assume that $p \notin C(co\{p,q\})$, then by nonemptiness of C there exists λ s.t. $\lambda p + (1-\lambda)q \in C(co\{p,q\})$, thus $\lambda p + (1-\lambda)q \in C(co\{p, \lambda p + (1-\lambda)q\})$ by IIA for constant acts, thus $q \in C(co\{p,q\})$ by independence. $\trianglerighteq_C$ is also transitive: Suppose by contradiction that $p \trianglerighteq_C q \trianglerighteq_C r \vartriangleright_C p$. The latter implies (using independence twice) that $(1-\gamma+\lambda\gamma)r + \gamma(1-\lambda)q \vartriangleright_C (1-\gamma)r + \gamma\lambda p + \gamma(1-\lambda)q$ for all $\gamma, \lambda \in (0,1]$. IIA (for constant acts) then yields $C(co\{p,q,r\}) \subseteq co\{q,r\}$, hence $\lambda r + (1-\lambda)q \in C(co\{p,q,r\})$ for some $\lambda \in (0,1)$. Independence, IIA, and $q \trianglerighteq_C r$ then imply $q \in C(co\{p,q,r\})$, after which IIA implies $p \in C(co\{p,q,r\})$, a contradiction. Being complete and transitive, $\trianglerighteq_C$ rationalizes C on the restriction of $\mathcal{F}$ to menus in $\Delta\mathcal{X}$. Furthermore, $\trianglerighteq_C$ is von Neumann-Morgenstern utility because Herstein and Milnor's [34] axioms are easy to verify. The remaining preliminaries apply with no or minimal modification; in particular, acts can be identified with utility acts.

Step 1: Defining an extension of $\succsim_C$. Define $\succsim_C$ as before, of course with the understanding that menus M are now convex. The argument that both $\succsim_C$ and any extension of it rationalize C on $\mathcal{M}_0$ is unchanged. Aymmetry of $\succ_C$ still goes through: Suppose by contradiction that $f \succ_C g$ is revealed in M and $g \succ_C f$ is revealed in N. Let $\Sigma_{M,N} \subseteq \Sigma$ be a finite partition of $\mathcal{S}$ s.t. any $f \in co(M \cup N)$ is constant on any $E \in \Sigma_{M,N}$. (This is possible because, while M and N are not finite, they are spanned by finite collections of finite acts.) Let $v = \min_{s \in \mathcal{S}, f \in M \cup N} u \circ f(s)$ and $L = co(\{f,g\} \cup \{p_{0E}p_v : E \in \Sigma_{M,N}\})$, then all elements of L are dominated by elements of M and N, hence $C(L) \cap \{f,g\} \neq \varnothing$ by INA and monotonicity, but then comparison of $C(L)$ with $C(co(M \cup L))$ or $C(co(N \cup L))$ must reveal a violation of INA. The argument for disjointness of $\succ_C$ and $\sim_C$ is similar. Again, $\succsim_C$ can be incomplete due to U being bounded from below, but this can be handled as before. Assume, therefore, that U is unbounded from below. Next, fix any event $E \in \Sigma \backslash \{\varnothing, \mathcal{S}\}$, then symmetry and convexity imply $p_{-1} \in C(co\{p_{-2E}p_0, p_{0E}p_{-2}\})$, hence some constant act (and then, by homotheticity, any constant act) is picked from some $M \in \mathcal{M}_0$, avoiding the problem from step 4 of lemma 2. However, $\succsim_C$ will fail to rank f and g if a proper mixture of them is strictly chosen over either. To circumvent this, $\succsim_C$ must be extended in a novel manner. Define the certainty equivalent $c(f)$ of any act $f \in \mathcal{F}_-$ as the constant act with utility value $u = \inf\{v \leq 0 : p_v \succ_C f\}$, with the convention that $c(f) = p_0$ if the set in this definition is empty. Let $f \succsim g \Leftrightarrow c(f) \geqq c(g)$. Then monotonicity and INA imply monotonicity of $\succsim$ (i.e., $f \geqq g \Rightarrow f \succsim g$), and homotheticity yields $c(\lambda f) = \lambda c(f)$. Also, suppose by contradiction that $f \succ_C c(f)$ and let M be the menu in which this is revealed. Continuity then implies the existence of $\lambda < 1$ s.t. $\lambda c(f) \notin C(co(M \cup \{\lambda c(f)\})$. By the same argument as in theorem 4, the c-betweenness axiom applies to any menu $M \in \mathcal{M}_0$. It now implies $C(co(M \cup \{\lambda c(f)\}) = C(M)$, thus $f \succ_C \lambda c(f)$, which together with monotonicity contradicts the definition of $c(f)$. Thus, $f \succ_C c(f)$ does not hold.

To wrap up this step, note $p \gg q \Rightarrow p \succ_C q$. Suppose otherwise, then there exist $p \gg q$ s.t. $p \succ_C q$ fails. Adding q to some menu M in which p is picked (existence of which was shown above), one can conclude $p \sim_C q$, but now homotheticity and monotonicity imply that all constant acts $p \ll p_0$, and in a second step all acts $f \ll p_0$, are indifferent, contradicting continuity by the same argument as in lemma 2, step 1.

Step 2: Symmetry of Certainty Equivalents. Fix any events $E, F \in \Sigma \backslash \{\varnothing, \mathcal{S}\}$ and constant acts $u, v \in \mathcal{F}_-$, then $c(u_E v) = c(u_F v)$. To prove this, it suffices to show $p \succ_C u_E v \Leftrightarrow p \succ_C u_F v$. Let $u > v$ w.l.o.g., initially assume that E and F are disjoint, and suppose $p \succ_C u_E v$. I show below that this preference is revealed in a menu M s.t. all acts $f \in M$ are constant on E and $\mathcal{S} \backslash E$. Thus, $p \succ_C u_F v$ follows by applying symmetry to M. The reverse implication follows analogously. If E and F are non-nested but overlap, one can similarly use symmetry to interchange the consequences of $E \cap F^c$ and $F \cap E^c$. If they are nested, say $E \subset F$, then use symmetry twice, using $p \succ_C u_{F^c} v$ as intermediate step.

It remains to show that $p \succ_C u_E v$ is revealed in a suitable menu M. To do so, restrict attention to acts that are constant on E and $\mathcal{S} \backslash E$, identify them with utility vectors $(u, v) \leqq (0, 0)$, and let z be the utility value of p, thus $p = (z, z)$. If $u + v < 2z$, the claim follows because $p \in C(co\{(2z, 0), (0, 2z), (u, v)\})$ by symmetry, convexity, INA, and monotonicity. Else, let $x = (zv - zu)/(v - z)$; this ensures that (z, z) is a convex combination of (u, v) and $(x, 0)$. Consider choice from $M = co\{(0, x), (u, v), (x, 0)\}$. $(z, z) \succ_C (u, v)$ ensures that $(u, v) \notin C(M)$; it remains to show that $(z, z) \in C(M)$. Suppose this fails, then $C(M) \cap co\{(u, v), (z, z)\} = \varnothing$ by c-betweenness. Note $x < 2z$, thus $(z, z) \in C(co\{(x, 0), (0, x), (z, z)\})$ through symmetry etc. as before, thus $C(M) \cap C(co\{(x, 0), (0, x), (z, z)\}) = \varnothing$ by INA. Also using monotonicity, it follows that $C(M)$ contains some $f \in co\{(0, x), (u, v)\}$. On the other hand, let the constant act q be s.t. (u, v) is a convex combination of $(0, x)$ and q, then $q \in C(co\{(0, x), q, (x, 0)\})$. Consider now the collection of menus $M_\lambda = co\{\{(0, x), (u, v), \lambda p + (1-\lambda) q, (x, 0)\} : \lambda \in [0, 1]\}$. Preceding arguments from this paragraph can be repeated to show that for any $\lambda \in [0,1]$, $C(M_\lambda)$ intersects either $co\{(0,x),(u,v)\}$ (and then, by INA, contains f) or $co\{(u,v), \lambda p + (1-\lambda)q\}$ (and then, by c-betweenness and given that it cannot contain (u,v), contains $\lambda p+(1-\lambda)q$). Now consider M_{λ^*}, where $\lambda^* = \sup\{\lambda : C(M_\lambda) \cap co\{(0,x),(u,v)\} = \varnothing\}$. Either of $f \notin C(M_{\lambda^*})$ and $\lambda^* p + (1-\lambda^*)q \notin C(M_{\lambda^*})$ would violate continuity, so both are chosen, but then (u,v) is chosen by convexity and monotonicity, a contradiction.

Step 3: Characterizing Certainty Equivalents. Consider a partition of $\mathcal{S}$ into three non-empty events $\{E_1, E_2, E_3\}$ and restrict attention to acts that are constant on each of $\{E_1, E_2, E_3\}$; these acts will be identified with utility vectors $(u,v,w) \leqq (0,0,0)$. The crucial claim is that $((u+v)/2,(u+v)/2,v) \succsim (u,v,v)$ for any $u > v$. To see this, it suffices to show that $p \succ_C ((u+v)/2,(u+v)/2,v) \Rightarrow p \succ_C (u,v,v)$. Thus, assume $p \succ_C ((u+v)/2,(u+v)/2,v)$. Let $M' = co\{(0,0,x),((u+v)/2,(u+v)/2,v),p,(x,x,0)\}$, then by step 2, $C(M') \cap co\,\{((u+v)/2,(u+v)/2,v),p\} = \{p\}$ for x low enough. Now consider $M'' = co(M' \cup \{(u,v,v),(v,u,v)\})$. M'' is invariant under exchange of the consequences of the first two events and M' contains all fixed points of such an exchange, hence symmetry and convexity jointly imply that $C(M'') \cap M' \neq \varnothing$, hence INA yields $C(M'') \cap \{((u+v)/2,(u+v)/2,v),p\} = \{p\}$. Now, if $(u,v,v) \in C(M'')$, then symmetry and convexity would jointly imply $((u+v)/2,(u+v)/2,v) \in C(M'')$, a contradiction. Hence, $p \succ_C (u,v,v)$ as required.

Recall that $(u,v,v) \sim (u,u,v)$ and $((u+v)/2,(u+v)/2,v) \sim ((u+v)/2,v,v)$ from step 2 and that $(u,u,v) \succsim ((u+v)/2,(u+v)/2,v)$ from step 1. Together with the preceding paragraph's finding, these imply $(u,v,v) \sim ((u+v)/2,v,v)$. Iterating this argument and using monotonicity, one finds that $(u,v,v) \sim (w,v,v)$ for any $w \in (v,0]$, which together with monotonicity and $p \gg q \Rightarrow p \succ_C q$ yields $c(u,v,v) \leqq (v,v,v)$. On the other hand, $(v,v,v) \succ_C (u,v,v)$ would violate monotonicity, hence $c(u,v,v) = (v,v,v)$.

This finding can be extended to general acts: Fix any act $f \in \mathcal{F}_-$, let $\underline{f}$ be the constant act with utility value $\min_{s \in \mathcal{S}} u \circ f(s)$ and let E be the event on which $f = \underline{f}$. Then $\underline{f}_E p_0 \succsim f \succsim \underline{f}$ by monotonicity, thus $\underline{f}_E p_0 \sim \underline{f}$ implies $f \sim \underline{f}$, thus $\succsim$ is priorless maximin.

Step 4: Extension. To see that $\succsim$ extends $\succsim_C$, suppose $f \succ_C g$ and let M be the menu in which this preference is observed. Consider choice from $N = co(M \cup \{c(f), c(g)\})$. Step 3 implies that $f \geqq c(f)$ and $g \geqq c(g)$, hence INA and monotonicity yield $f \in C(N)$, but now $c(f) \in C(N)$ because $f \succ_C c(f)$ was excluded. On the other hand, $g \notin C(N) \Rightarrow c(g) \notin C(N)$ by monotonicity, hence $c(f) \gg c(g)$. The argument for $f \sim_C g \Rightarrow f \sim g$ is similar.

Step 5: Characterizing C in terms of $\succsim$. See step 2 of the proof of lemma 2.

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