

Chapter 8

COOPER PAIRS

Superconductivity arises from a very complicated set of interactions between the electrons in a metal. I would like to consider a very simplified model of the interaction of a pair of electrons in a metal, since on the one hand, it is an instructive example of how quantum mechanics works, and on the other hand, the main features of this model are at the starting point of the modern theory of superconductivity.

If we neglect all effects of the crystal structure, then a metal of volume V can be looked upon simply as a box filled with electrons. If we neglect all interactions between electrons, the normalized energy eigenstates, using periodic boundary conditions, are

$$\psi(\mathbf{r}) = (V)^{-1/2} e^{i\mathbf{k} \cdot \mathbf{r}/\hbar} \quad (8-1)$$

where $\epsilon_{\mathbf{k}} = k^2/2m$. The possible \mathbf{k} vectors in a cubic box of side L , are given by

$$k_x = \frac{2\pi n_x \hbar}{L}, \quad k_y = \frac{2\pi n_y \hbar}{L}, \quad k_z = \frac{2\pi n_z \hbar}{L} \quad (8-2)$$

where n_x , n_y , and n_z are integers ranging from $-\infty$ to ∞ .

When we fill up the box with electrons we can put two electrons in each state (two for spin). The configuration of lowest total energy for N electrons will have the states filled up to a certain maximum value, k_f , the *Fermi momentum*, which is given by

$$N = \sum_{\mathbf{k} < k_f} 2, \quad (8-3)$$

where the sum is over all \mathbf{k} with $k < k_f$. The filled \mathbf{k} states form

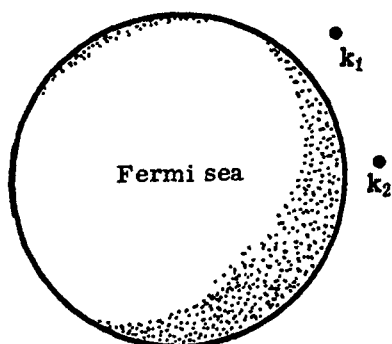


Fig. 8-1

Two noninteracting electrons outside the Fermi sea.

a sphere, called the *Fermi sea* [Fig. 8-1]. For macroscopic L and N , the k vectors are spaced closely enough so that we can replace the sum over k by an integral. Since the interval between k_x values is $2\pi\hbar/L$,

$$\sum_{k_x} \rightarrow \frac{L}{2\pi\hbar} \int dk_x$$

and

$$\sum_k \rightarrow V \int \frac{d^3k}{(2\pi\hbar)^3}, \quad (8-4)$$

where $V = L^3$. There are $V/(2\pi\hbar)^3$ states per unit volume in momentum space. From (8-3) we then find that k_f is given by

$$k_f = (3\pi^2 n)^{1/3} \hbar \quad (8-5)$$

where $n = N/V$ is the density of particles; k_f is an *intensive* parameter.

The model we want to consider is that of two electrons just outside the surface of the Fermi sea (the Fermi surface) interacting through a weak attractive force. We shall neglect all interactions between the electrons in the Fermi sea, and between the electrons in the Fermi sea and the pair. This is where this model differs from a model of superconductivity. We shall also assume that the electrons

have opposite spin values, one up and the other down, and therefore the exclusion principle does not prevent them from being in the same spatial state simultaneously. Our problem shall be to find the energy eigenstates of the interacting pair of electrons.

If there is no interaction between the two electrons, then the energy eigenstates of the pair are simply of the form

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \frac{e^{i\mathbf{k}_1 \cdot \mathbf{r}_1/\hbar}}{\sqrt{V}} \frac{e^{i\mathbf{k}_2 \cdot \mathbf{r}_2/\hbar}}{\sqrt{V}} e^{-i(\epsilon_{\mathbf{k}_1} + \epsilon_{\mathbf{k}_2})t/\hbar}. \quad (8-6)$$

Now if the electrons interact with each other, then (8-6) will no longer be a stationary state, or energy eigenstate, since as the particles scatter from each other they keep changing their momentum values. The states of the interacting pair will be of the form

$$\psi(\mathbf{r}_1, \mathbf{r}_2, t) = \sum_{\mathbf{k}_1 \mathbf{k}_2} a_{\mathbf{k}_1 \mathbf{k}_2}(t) \frac{e^{i\mathbf{k}_1 \cdot \mathbf{r}_1/\hbar}}{\sqrt{V}} \frac{e^{i\mathbf{k}_2 \cdot \mathbf{r}_2/\hbar}}{\sqrt{V}} \quad (8-7)$$

and for an energy eigenstate

$$a_{\mathbf{k}_1 \mathbf{k}_2}(t) = e^{-iEt/\hbar} a_{\mathbf{k}_1 \mathbf{k}_2}; \quad (8-8)$$

E is the total energy of the pair. Equation (8-7) expresses ψ as a double Fourier series in \mathbf{r}_1 and \mathbf{r}_2 . The amplitude $a_{\mathbf{k}_1 \mathbf{k}_2}(t)$ is the amplitude for finding particle 1 with momentum \mathbf{k}_1 and 2 with momentum \mathbf{k}_2 . Because all the states inside the Fermi sea are already filled, the amplitude for finding either particle in a state inside the Fermi sea must be zero, that is, $a_{\mathbf{k}_1 \mathbf{k}_2}$ is zero unless both \mathbf{k}_1 and \mathbf{k}_2 are greater than k_F .

How does the amplitude $a_{\mathbf{k}_1 \mathbf{k}_2}(t)$ change in time? If there is no interaction then because (8-6) is a stationary state,

$$i\hbar \frac{\partial}{\partial t} a_{\mathbf{k}_1 \mathbf{k}_2}(t) = (\epsilon_{\mathbf{k}_1} + \epsilon_{\mathbf{k}_2}) a_{\mathbf{k}_1 \mathbf{k}_2}(t); \quad (8-9)$$

only the phase of the amplitude changes. When the particles interact, then if at one instant they have momenta \mathbf{k}_1 and \mathbf{k}_2 , at a slightly later instant they will have an amplitude for having different momenta \mathbf{k}_1' and \mathbf{k}_2' , because they can scatter from each other.

Thus we expect a change in the amplitude $a_{\mathbf{k}_1 \mathbf{k}_2}(t)$ due to pairs with \mathbf{k}_1' and \mathbf{k}_2' scattering into $\mathbf{k}_1, \mathbf{k}_2$ [Fig. 8-2]; this term will be

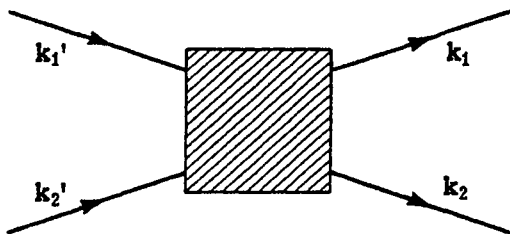


Fig. 8-2
Scattering of a pair from k_1', k_2' , to k_1, k_2 .

proportional to the amplitude $a_{k_1', k_2'}$ for the particles to have momenta k_1', k_2' . This process will add a term on the right side of (8-9) of the form $\sum_{k_1', k_2'} \langle k_1 k_2 | \mathcal{V} | k_1' k_2' \rangle a_{k_1', k_2'}$. The quantity $\langle k_1 k_2 | \mathcal{V} | k_1' k_2' \rangle$ is called the matrix element for scattering from k_1', k_2' to k_1, k_2 . With this term the Schrödinger equation for the rate of change of the amplitude becomes

$$i\hbar \frac{\partial}{\partial t} a_{k_1 k_2}(t) = (\epsilon_{k_1} + \epsilon_{k_2}) a_{k_1 k_2}(t) + \sum_{k_1', k_2'} \langle k_1 k_2 | \mathcal{V} | k_1' k_2' \rangle a_{k_1', k_2'}(t). \quad (8-10)$$

It looks as if we've left out a change in the amplitude due to particles with k_1, k_2 scattering to other states k_1', k_2' . This term, being proportional to a_{k_1, k_2} , has the same structure as the "diagonal" term $\langle k_1 k_2 | \mathcal{V} | k_1 k_2 \rangle a_{k_1, k_2}$ in (8-10), so we can regard it as being already included in (8-10).

One can regard the amplitudes $\langle k_1 k_2 | \mathcal{V} | k_1' k_2' \rangle$ as being the matrix elements of an interaction operator \mathcal{V} in the " k_1, k_2 basis." In a very complicated system, the matrix elements $\langle k_1 k_2 | \mathcal{V} | k_1' k_2' \rangle$ are hard to determine; usually one has to make educated guesses and see how well the results agree with experiment.

One immediate requirement on the matrix elements is that the total probability $\sum_{k_1, k_2} |a_{k_1, k_2}(t)|^2$ not change in time. This implies, as usual, that

$$\langle k_1 k_2 | \mathcal{V} | k_1' k_2' \rangle^* = \langle k_1' k_2' | \mathcal{V} | k_1 k_2 \rangle; \quad (8-11)$$

\mathcal{V} must be a Hermitian matrix.

It is illuminating to write (8-10) in terms of the wave function $\psi(\mathbf{r}_1, \mathbf{r}_2, t)$. The result is

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}_1, \mathbf{r}_2, t) = \left(-\frac{\hbar^2}{2m} \nabla_1^2 - \frac{\hbar^2}{2m} \nabla_2^2 \right) \psi(\mathbf{r}_1, \mathbf{r}_2, t) \\ + \int d\mathbf{r}_1' d\mathbf{r}_2' \langle \mathbf{r}_1 \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1' \mathbf{r}_2' \rangle \psi(\mathbf{r}_1', \mathbf{r}_2', t), \quad (8-12)$$

where

$$\langle \mathbf{r}_1 \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1' \mathbf{r}_2' \rangle = \frac{1}{V^2} \sum_{\substack{\mathbf{k}_1, \mathbf{k}_2 \\ \mathbf{k}_1', \mathbf{k}_2'}} e^{i(\mathbf{k}_1 \cdot \mathbf{r}_1 + \mathbf{k}_2 \cdot \mathbf{r}_2)/\hbar} \langle \mathbf{k}_1 \mathbf{k}_2 | \mathcal{V} | \mathbf{k}_1' \mathbf{k}_2' \rangle \\ \times e^{(-i\mathbf{k}_1' \cdot \mathbf{r}_1' - i\mathbf{k}_2' \cdot \mathbf{r}_2')/\hbar} \quad (8-13)$$

Equation (8-12) is in the form of a Schrödinger equation, only with a *nonlocal* potential $\langle \mathbf{r}_1 \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1' \mathbf{r}_2' \rangle$. The rate of change of the amplitude for the particles being at $\mathbf{r}_1, \mathbf{r}_2$ depends on the amplitude for the particles being at far away points as well as close points. If the interaction can be represented simply by a potential $v(\mathbf{r}_1 - \mathbf{r}_2)$, as in the hydrogen atom, then

$$\langle \mathbf{r}_1 \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1' \mathbf{r}_2' \rangle = \delta(\mathbf{r}_1 - \mathbf{r}_1') \delta(\mathbf{r}_2 - \mathbf{r}_2') v(\mathbf{r}_1 - \mathbf{r}_2). \quad (8-14)$$

In this case

$$\langle \mathbf{k}_1 \mathbf{k}_2 | \mathcal{V} | \mathbf{k}_1' \mathbf{k}_2' \rangle = \frac{1}{V} v(\mathbf{k} - \mathbf{k}') \delta_{\mathbf{K}, \mathbf{K}'} \quad (8-15)$$

where $\mathbf{k} = (\mathbf{k}_1 - \mathbf{k}_2)/2$, $\mathbf{k}' = (\mathbf{k}_1' - \mathbf{k}_2')/2$, $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$ and $\mathbf{K}' = \mathbf{k}_1' + \mathbf{k}_2'$; also $v(\mathbf{q}) = \int d^3r e^{-i\mathbf{q} \cdot \mathbf{r}} v(\mathbf{r})$. The δ symbol guarantees that the total momentum of the pair is conserved in the interaction.

Generally, if the total momentum of the pair of particles is conserved in the interaction the matrix element connects only states with the same total momentum and thus it has the form

$$\langle \mathbf{k}_1 \mathbf{k}_2 | \mathcal{V} | \mathbf{k}_1' \mathbf{k}_2' \rangle = \mathcal{V}_{\mathbf{k}, \mathbf{k}'}(\mathbf{K}) \delta_{\mathbf{K}, \mathbf{K}'}. \quad (8-16)$$

Then the nonlocal potential has the form

$$\langle \mathbf{r}_1 \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1' \mathbf{r}_2' \rangle = \langle \mathbf{r} | \mathcal{V}(\mathbf{R} - \mathbf{R}') | \mathbf{r}' \rangle, \quad (8-17)$$

where $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, $\mathbf{R}' = (\mathbf{r}_1' + \mathbf{r}_2')/2$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}' = \mathbf{r}_1' - \mathbf{r}_2'$ are the center-of-mass and relative coordinates.

In an energy eigenstate, when total momentum is conserved

$$a_{\mathbf{k}_1, \mathbf{k}_2}(t) = a_{\mathbf{k}}(\mathbf{K}) e^{-iEt/\hbar}, \quad (8-18)$$

and (8-10) becomes

$$(E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}) a_{\mathbf{k}}(\mathbf{K}) = \sum_{\mathbf{k}'} \gamma_{\mathbf{k}, \mathbf{k}'}(\mathbf{K}) a_{\mathbf{k}'}(\mathbf{K}). \quad (8-19)$$

The total momentum appears in this equation only as a parameter. For a general interaction $\gamma_{\mathbf{k}, \mathbf{k}'}(\mathbf{K})$, Eq. (8-19) is very hard to solve.

In a metal one has a repulsive Coulomb interaction between the electrons, together with an attractive interaction between the electrons and ions. Now when an electron moves, it tends to pull the ions toward it. Of course, the ions don't get very far because they are very heavy and are bound to the vicinity of their lattice sites. However, when an ion moves toward the electron, the other electrons tend to follow the ion, and therefore the first electron. This means that because of the presence of the ions, there is some tendency for electrons to attract each other. The total interaction between any two electrons is a composite of this attraction plus the Coulomb repulsion. In some metals, this total interaction is attractive for electrons near the Fermi surface. To a first approximation we can represent this interaction by

$$\gamma_{\mathbf{k}, \mathbf{k}'}(\mathbf{K}) = \begin{cases} -\frac{v_0}{V}: & k_f < k_1, k_2, k_1', k_2' < k_a \\ 0: & \text{otherwise} \end{cases} \quad (8-20)$$

where v_0 is positive, and k_a is a momentum slightly greater than k_f .

With this interaction, the Schrödinger equation becomes

$$(E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}) a_{\mathbf{k}}(\mathbf{K}) = -\frac{v_0}{V} \sum_{\mathbf{k}'}' a_{\mathbf{k}'}(\mathbf{K}), \quad (8-21)$$

where the prime on the sum indicates that only values of \mathbf{k}' between k_f and k_a , and such that $k_f < |(\mathbf{K}/2) \pm \mathbf{k}'| < k_a$ are to be summed over.¹ To solve (8-21) we divide both sides by $E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}$ and sum over the allowed values of \mathbf{k} . Then

¹ For an *s*-state, for which $a_{\mathbf{k}}(\mathbf{K})$ is independent of the direction of \mathbf{k} , the sum over \mathbf{k}' cannot vanish. Some $a_{\mathbf{k}}(\mathbf{K})$ must be nonzero else $\psi \equiv 0$. Then if the sum vanished, Eq. (8-21) would imply that for this \mathbf{k} , $E = \varepsilon_{\mathbf{k}_1} + \varepsilon_{\mathbf{k}_2} = (k^2/m) + (K^2/4m)$. However only one \mathbf{k} can satisfy this condition, so that $a_{\mathbf{k}}(\mathbf{K})$ can be nonzero only for one \mathbf{k} . Thus the only non-zero terms in the sum are all equal, whereupon the sum over \mathbf{k}' must be nonzero.

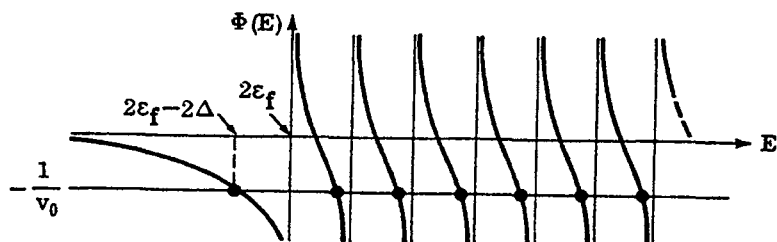


Fig. 8-3

Graphical solution of Eq. (8-24).

$$\sum_{\mathbf{k}}' a_{\mathbf{k}}(\mathbf{K}) = -\frac{v_0}{V} \sum_{\mathbf{k}}' \frac{1}{E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}} \sum_{\mathbf{k}'}' a_{\mathbf{k}'}(\mathbf{K}). \quad (8-22)$$

Canceling the sum over $a_{\mathbf{k}}$ from both sides we find a condition to determine the possible eigenvalues E :

$$1 = -\frac{v_0}{V} \sum_{\mathbf{k}}' \frac{1}{E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}}. \quad (8-23)$$

The nature of the solutions can be seen graphically. Let

$$\Phi(E) = \frac{1}{V} \sum_{\mathbf{k}}' \frac{1}{E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}}. \quad (8-24)$$

The eigenvalue condition is thus $\Phi(E) = -1/v_0$. $\Phi(E)$ has a pole at each possible energy, $\varepsilon_{\mathbf{k}_1} + \varepsilon_{\mathbf{k}_2}$, of a noninteracting pair of electrons outside the Fermi sea, with total momentum \mathbf{K} , and k_1 and k_2 between k_f and k_a ; these energies are at least $2\varepsilon_f$, where $\varepsilon_f = k_f^2/2m$. Thus $\Phi(E)$ looks as shown in Fig. 8-3. We see that $\Phi(E)$ intersects $-1/v_0$ at many places above the minimum energy of a noninteracting pair; these energies correspond to states qualitatively like the noninteracting states (8-6). Also, for v_0 positive, there is always one intersection at an energy, E_b , below the minimum. This means that the interaction has produced a "bound state" of the two electrons; this new state is quite different from the noninteracting states (8-6).

Let us solve for E_b in the case $\mathbf{K} = 0$. Then $\mathbf{k}_1 = \mathbf{k}$, $\mathbf{k}_2 = -\mathbf{k}$, and the restriction that k_1 and k_2 be outside the Fermi surface is trivial to handle; replacing the sum by an integral we have

$$\Phi(E) = \int_{k_f}^{k_a} \frac{d^3k}{(2\pi\hbar)^3} \frac{1}{E - 2\varepsilon_k}, \quad (8-25)$$

$$= \frac{m}{2\pi^2\hbar^3} \int_{\varepsilon_f}^{\varepsilon_a} k \, d\varepsilon_k \frac{1}{E - 2\varepsilon_k}, \quad (8-26)$$

where $\varepsilon_a = k_a^2/2m$. For ε_a close to ε_f we can replace the k inside the integrand by k_f , and we find, for $E < 2\varepsilon_f$,

$$\Phi(E) = -\frac{N(0)}{2} \ln \left| \frac{2\varepsilon_a - E}{2\varepsilon_f - E} \right|, \quad (8-27)$$

where

$$N(0) = \frac{mk_f}{2\pi^2\hbar^3} = \int \frac{d^3k}{(2\pi\hbar)^3} \delta(\varepsilon_k - \varepsilon_f) \quad (8-28)$$

is the density of states at the Fermi surface. Equating $\Phi(E)$ to $-1/v_0$, and writing

$$E_b = 2\varepsilon_f - 2\Delta$$

we find

$$\Delta = \frac{\varepsilon_a - \varepsilon_f}{e^{2/v_0 N(0)} - 1} \quad (8-29)$$

The energy $\varepsilon_a - \varepsilon_f$ is on the order of the Debye energy, ω_D , which is \hbar times the maximum frequency of a lattice oscillation in the metal. Typically $\omega_D/\varepsilon_f \sim 1/100$, and $v_0 N(0) \sim 1/4$. Thus the binding energy per electron in this bound state is

$$\Delta \approx \omega_D e^{-2/v_0 N(0)}. \quad (8-30)$$

Notice that Δ is a highly nonanalytic function of the parameter v_0 . This mechanism for binding electrons together, which is called "pairing," was discovered by Cooper; electrons in such bound states are called Cooper pairs.²

The number of possible k values allowed in (8-24) drops sharply as K becomes different from zero. The poles of $\Phi(E)$ are at the energy values of the possible noninteracting pairs but because there

² L.N. Cooper *Phys. Rev.* 104, 1189 (1956).

are fewer \mathbf{k} values allowed, $\Phi(E)$ is smaller in magnitude for E less than the minimum singularity. This means that for $\mathbf{K} \neq 0$ the intersection of $\Phi(E)$ with $-1/v_0$ lies closer to the minimum noninteracting energy value of the pair than for $\mathbf{K} = 0$, and thus the binding energy of a bound pair decreases, rapidly in fact, with increasing \mathbf{K} . The point is that the fewer the number of states connected together by the attractive interaction $\langle \mathbf{k}_1 \mathbf{k}_2 | \mathcal{V} | \mathbf{k}_1' \mathbf{k}_2' \rangle$, the smaller will be the binding energies that result. The biggest binding energy occurs for electrons diametrically opposite each other around the Fermi sea.

To find the wave function of a Cooper pair we notice that from (8-21)

$$a_{\mathbf{k}}(\mathbf{K}) = \frac{1}{E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}} \times \text{constant} \quad (8-31)$$

Thus, from (8-7), keeping \mathbf{K} fixed in the sum

$$\psi(\mathbf{r}_1, \mathbf{r}_2) \sim e^{i\mathbf{K} \cdot (\mathbf{r}_1 + \mathbf{r}_2)/2\hbar} \frac{1}{V} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)/\hbar}}{E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}}. \quad (8-32)$$

The relative wave function $\varphi(\mathbf{r})$ is thus

$$\varphi(\mathbf{r}) \sim \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \frac{e^{i\mathbf{k} \cdot \mathbf{r}/\hbar}}{E - \varepsilon_{\mathbf{k}_1} - \varepsilon_{\mathbf{k}_2}}. \quad (8-33)$$

For the $\mathbf{K} = 0$ bound pair $\varphi(\mathbf{r})$ behaves roughly as $(1/r) \sin(k_f r/\hbar)$ times a slowly varying function of r similar to $\sin(m\Delta r/2k_f\hbar)$. The length $\xi = 2k_f\hbar/m\Delta$ is essentially the size of bound pair. The wave function $\varphi(\mathbf{r})$ is spherically symmetric, and therefore the pair has angular momentum zero; it is in an s -state.

To explain superconductivity one must take into account the interactions between *all* the electrons at the same time. Then one finds that all the electrons at the Fermi surface form pairs with the same total momentum. When this momentum is different from zero there is a net current — a supercurrent. In order to slow down this current one can't simply slow down the electrons one at a time, as in a normal conductor, because they are all paired together to the same total momentum; one has to slow down all the electrons at the same time. Because this is extraordinarily difficult to do, supercurrents persist indefinitely.³

³ See, e.g., J.R. Schrieffer, *Theory of Superconductivity* [W.A. Benjamin, New York, 1964].

A few final comments about the nonlocal potential $\langle \mathbf{r}_1 \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1' \mathbf{r}_2' \rangle$: If we assume that the relative angular momentum of the pair of particles is conserved in their interaction, then if in the Schrödinger equation (8-12) $\psi(\mathbf{r}_1, \mathbf{r}_2, t)$ is an eigenstate of angular momentum, so must be the term

$$\int d^3\mathbf{r}_1' d^3\mathbf{r}_2' \langle \mathbf{r}_1 \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1' \mathbf{r}_2' \rangle \psi(\mathbf{r}_1', \mathbf{r}_2', t),$$

in order that the angular momentum of the pair be constant in time. This implies that $\langle \mathbf{r}_1 \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1' \mathbf{r}_2' \rangle$ must be of the form, when total momentum is also conserved

$$\langle \mathbf{r}_1, \mathbf{r}_2 | \mathcal{V} | \mathbf{r}_1', \mathbf{r}_2' \rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(\Omega_{\mathbf{r}}) Y_{lm}(\Omega_{\mathbf{r}'}) \mathcal{V}_l(\mathbf{R} - \mathbf{R}', |\mathbf{r}|, |\mathbf{r}'|) \quad (8-34)$$

where $\Omega_{\mathbf{r}}$ denotes the angles of the vector \mathbf{r} . Then Fourier transforming, and using formula (6-85) we see that $\mathcal{V}_{\mathbf{k}, \mathbf{k}'}(\mathbf{K})$ must be of the form

$$\mathcal{V}_{\mathbf{k}, \mathbf{k}'}(\mathbf{K}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(\Omega_{\mathbf{k}}) Y_{lm}(\Omega_{\mathbf{k}'}) \mathcal{V}_{l, \mathbf{k}, \mathbf{k}'}(\mathbf{K}). \quad (8-35)$$

$\mathcal{V}_{l, \mathbf{k}, \mathbf{k}'}(\mathbf{K})$ is the matrix element for the scattering from a state with magnitude of relative momentum k' and relative angular momentum l to one of relative momentum k and angular momentum l . Looking back at the interaction (8-20) we see that it is an interaction only between particles in relative s -states, which explains why the wave function of the Cooper pair also was an s -state.

In addition to electrons in metals, pairing takes place in liquid He^3 below 0.0027°K , in p -states. Nucleons outside closed shells in nuclei, as well as bulk nuclear matter in neutron stars, can also undergo pairing.

PROBLEMS

1. Calculate $\langle r^2 \rangle$ for the $K = 0$ bound pair.
2. To see the role played by the Fermi sea in the Cooper pair problem, suppose that $k_f = 0$. What is then the exact condition on v_0 that there be a bound state ($E < 0$) for $K = 0$?

3. Estimate as a function of K the volume of \mathbf{k} -space entering the sum (8-24) for $K \neq 0$.
4. Find the possible bound state energies, and eigenfunctions for a Cooper pair with total momentum zero, if the potential is attractive and constant in d -states when both particles are in a thin shell about the Fermi surface, and otherwise zero.