Two examples given in class:

1) If we roll four dice, what is the probability of at least one six?

a) Consider the complement problem, there is a 5/6 probability of not rolling a six for any given die, and since the four dice are independent, the probability of not rolling a six is $(5/6)^4 = 5^4/6^4 = 625/1296$. The probability of rolling at least one six is therefore $1 - 625/1296 = 671/1296 \approx .517$.

b) Alternatively, recall that the number of ways of choosing r objects from a collection of N is $\binom{N}{r} = N!/r!(N-r)!$.

Any of the four dice can be the one that comes up six, and the other three don't, so the number of ways that exactly one of the four dice is six is $\binom{4}{1} \cdot 5^3 = 4 \cdot 5^3 = 500$

exactly two sixes: $\binom{4}{2} \cdot 5^2 = (4 \cdot 3/2) \cdot 5^2 = 150$

exactly three sixes: $\binom{4}{3} \cdot 5 = 4 \cdot 5 = 20$

exactly four sixes: $\binom{4}{4} = 1$

The total number of possibilities is $500 + 150 + 20 + 1 = 671$, and hence the probability is $671/6^4$, in agreement with the above.

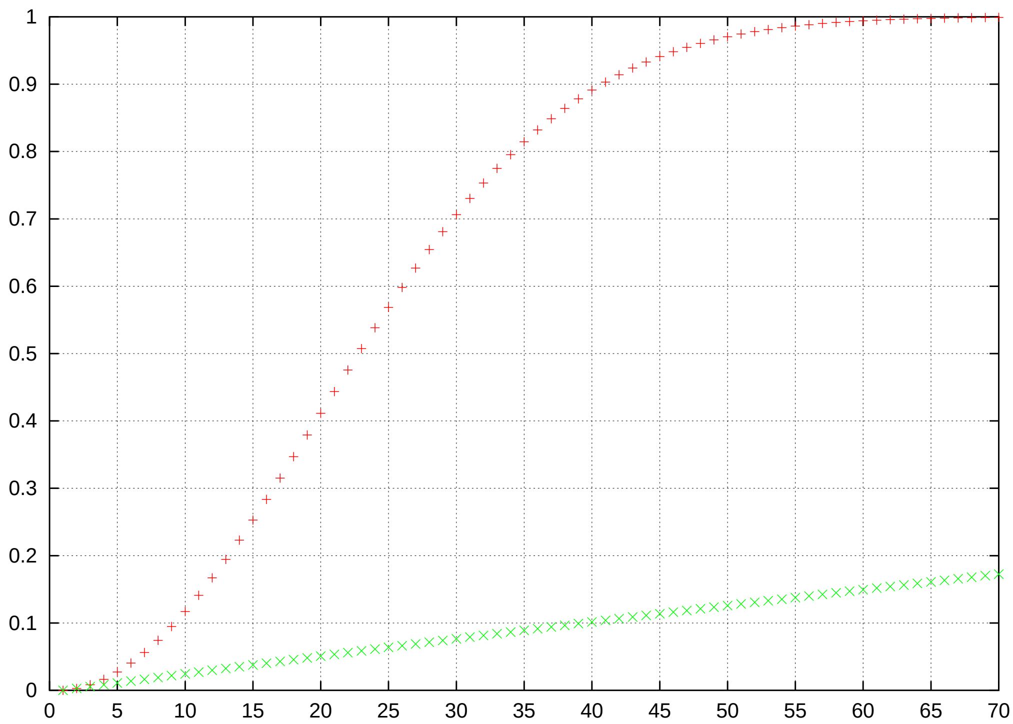
2) a) What is the probability that in a group of N people, at least two have the same birthday?
(Simplifications: assume no leap years, and assume that all birthdays are equally likely.)

Again consider the complement problem, the probability that no two birthdays coincide. The total number of possibilities with no coincidences is $365 \cdot 364 \cdot \ldots \cdot (366 - n)$ (i.e., n factors each successive one with one fewer choice of day). The total number of possibilities for n choices of birthdays is 365^n, so the probability of no coincidences is $365 \cdot 364 \cdot \ldots \cdot (366 - n)/365^n$. The probability that at least two coincide is therefore $1 - 365 \cdot 364 \cdot \ldots \cdot (366 - n)/365^n$.

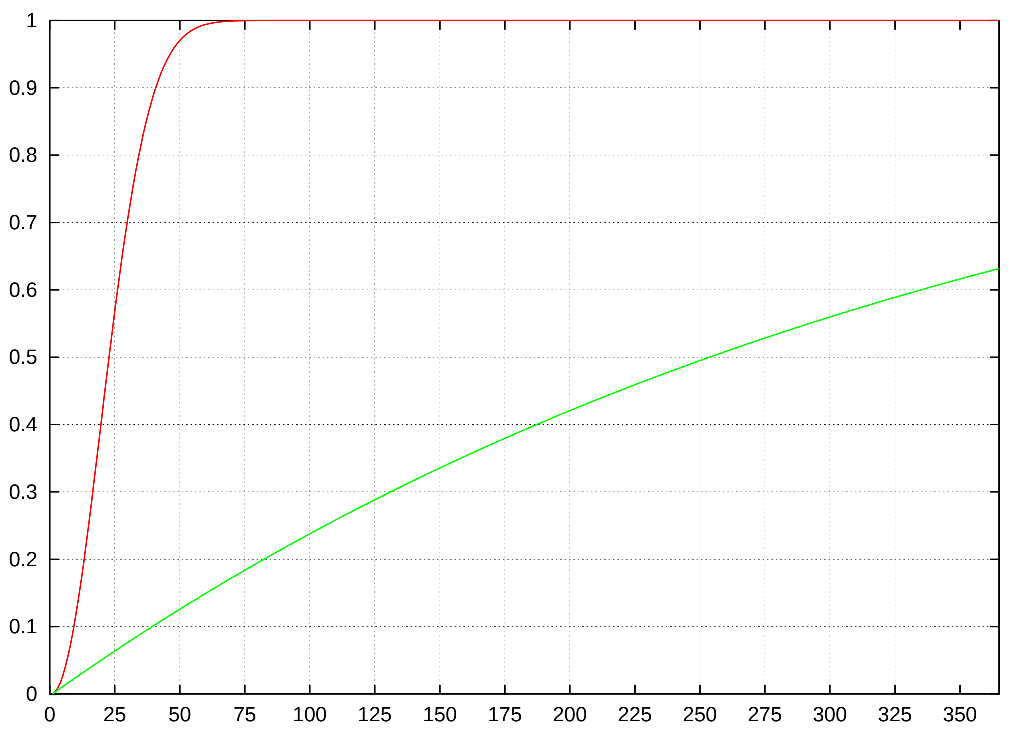
This probability is rapidly increasing as a function of n and turns out to be greater than .5 for $n = 23$. (See graph on next page).

b) In a group of 23 people, what it the probability that at least one person has a birthday coincident specifically with *yours*?

In this case, we first calculate the probability that none of the 22 others (again under the above simplifications) has a birthday coincident with a given day: $(364/365)^{22}$. The probability that at least one coincides with that day is therefore $1 - (364/365)^{22} \approx .059$, so a roughly 6% chance. This probably increases more slowly as a function of the size of the group (see graphs next page).



Red (upper): The probability $1 - \frac{365!}{(365-n)!365^n}$ that at least two birthdays coincide within a group of n people, as function of n.
Green (lower): The probability $1 - \left(\frac{364}{365}\right)^{n-1}$ of a birthday coinciding with yours within a group of n people including you.



Same as above, expanded to show n up to 365. The probability for the lower case at $n = 365$ is roughly $1 - 1/e \approx .632$.